- Review of Hyperelasticity

existence of strain energy potential

$\xrightarrow{\text{implies}}$ path-independent, reversible, nondissipative $\Big\}$ material deformation process

$$\psi \longrightarrow S_{IJ} = 2\frac{\partial \psi(\underset{\sim}{C})}{\partial C_{IJ}} \longrightarrow C^{SE}_{IJKL} = 4\frac{\partial^2 \psi(\underset{\sim}{C})}{\partial C_{IJ}\partial C_{KL}}$$

$\underset{\sim}{\sigma} = J^{-1}\underset{\sim}{F}\,\underset{\sim}{S}\,\underset{\sim}{F}^T$ (push-forward)

$$\boxed{\sigma_{ij} = J^{-1}F_{iI}F_{jJ}S_{IJ}}$$

Cauchy stress

$\underset{\sim}{\tau}^{\nabla c} = \underset{\sim}{F}\,\dot{\underset{\sim}{S}}\,\underset{\sim}{F}^T$

$\underset{\sim}{\sigma}^{\nabla T} = J^{-1}\underset{\sim}{\tau}^{\nabla c}$

$= J^{-1}\phi_*[\underset{\sim}{C}^{SE}] : \underset{\sim}{D}$

$$\boxed{C^{\sigma T}_{ijkl} = J^{-1}\phi_*[\underset{\sim}{C}^{SE}]}$$

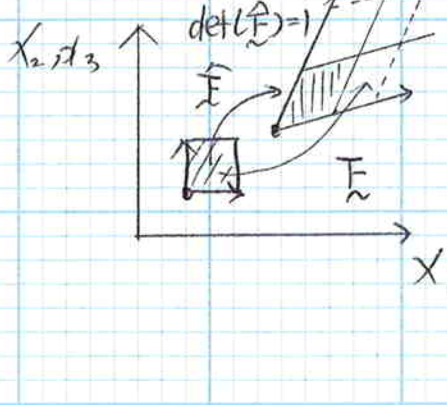
tangent moduli that relates Truesdell rate of Cauchy stress w/ rate-of-deformation.

omit

Recall that $\underset{\sim}{F} = \underset{\sim}{F}^{vol}\hat{\underset{\sim}{F}}$ $\qquad \hat{\underset{\sim}{F}} = \underset{\sim}{F}^{dev}$.

$\underset{\sim}{F}^{vol} = J^{\frac{1}{3}}$

$\hat{\underset{\sim}{F}} = \underset{\sim}{F}^{dev} = J^{-1/3}\underset{\sim}{F}$ $\quad \det(\hat{F}) = 1$ $\quad$ distortional part of (deviatoric) deformation



$$\underset{\sim}{\sigma}^{hyd} : \underset{\sim}{D}^{dev} = \underset{\sim}{\sigma}^{dev} : \underset{\sim}{D}^{hyd} = 0$$

$$\underset{\sim}{S}^{hyd} : \dot{\underset{\sim}{E}}^{dev} = \underset{\sim}{S}^{dev} : \dot{\underset{\sim}{E}}^{vol} = 0$$

in a sense of orthogonality.

⊙ Isotropic Elasticity in Principal Directions

recall $\psi(\underset{\sim}{C}(\underset{\sim}{X}), \underset{\sim}{X}) = \psi(I_C, II_C, III_C, \underset{\sim}{X})$

PK2 stress

$$\underset{\sim}{S} = 2\frac{\partial \psi}{\partial \underset{\sim}{C}} = 2\frac{\partial \psi}{\partial I_C}\frac{\partial I_C}{\partial \underset{\sim}{C}} + 2\frac{\partial \psi}{\partial II_C}\frac{\partial II_C}{\partial C} + 2\frac{\partial \psi}{\partial III_C}\frac{\partial III_C}{\partial C}$$

$$\frac{\partial I_C}{\partial \underset{\sim}{C}} = \frac{\partial C_{KK}}{\partial C_{IJ}} = \delta_{KI}\delta_{KJ} = \delta_{IJ} = \underset{\sim}{I}$$

$$\frac{\partial II_C}{\partial \underset{\sim}{C}} = \frac{\partial C_{KL}C_{KL}}{\partial C_{IJ}} = \delta_{KI}\delta_{LJ}C_{KL} + C_{KL}\delta_{KI}\delta_{LJ}$$
$$= C_{IJ} + C_{IJ} = 2C_{IJ} = 2\underset{\sim}{C}$$

$$\frac{\partial III_C}{\partial \underset{\sim}{C}} = J^2 \underset{\sim}{C}^{-1} \qquad D III_C[\Delta \underset{\sim}{C}] = \boxed{\det(\underset{\sim}{C})(\underset{\sim}{C}^{-1}} : \Delta \underset{\sim}{C})$$
$$= \boxed{\frac{\partial III_C}{\partial \underset{\sim}{C}}} : \Delta \underset{\sim}{C}$$

$$\underset{\sim}{S} = 2\psi_I \underset{\sim}{I} + 4\psi_{II}\underset{\sim}{C} + 2J^2\psi_{III}\underset{\sim}{C}^{-1}$$

let's write constitutive equation in terms of the stretches $\lambda_1, \lambda_2, \lambda_3$ and its principal directions N_1, N_2 and N_3.

Knowing

$$\underset{\sim}{I} = \sum_{\alpha=1}^{3} N_\alpha \otimes N_\alpha$$

$$\underset{\sim}{C} = \sum_{\alpha=1}^{3} \lambda_\alpha^2 N_\alpha \otimes N_\alpha$$

summation doesn't apply.

$$\underset{\sim}{C}^{-1} = \sum_{\alpha=1}^{3} \lambda_\alpha^{-2} N_\alpha \otimes N_\alpha$$

$$\underset{\sim}{S} = \sum_{\alpha=1}^{3}(2\psi_I + 4\psi_{II}\lambda_\alpha^2 + 2III_C\psi_{III}\lambda_\alpha^{-2}) N_\alpha \otimes N_\alpha \quad \text{— (1)}$$

$\frac{\partial \psi_I}{\partial I_{\underset{\sim}{C}}}$ $\quad \frac{\partial \psi}{\partial II_C}$ $\quad \frac{\partial \psi}{\partial III_C}$

should be changed to differentiation w.r.t. stretches

Recalling

$$I_C = \lambda_1^2 + \lambda_2^2 + \lambda_3^2 = \mathrm{tr}(\underset{\sim}{C}) \qquad 1 = \frac{\partial I_C}{\partial \lambda_\alpha^2}$$

$$II_C = \lambda_1^4 + \lambda_2^4 + \lambda_3^4 = \mathrm{tr}(\underset{\sim}{C}^2) = \underset{\sim}{C} : \underset{\sim}{C} \Rightarrow \frac{\partial II_C}{\partial \lambda_\alpha^2} = 2\lambda_\alpha^2$$

$$III_C = \lambda_1^2 \lambda_2^2 \lambda_3^2 = \det(\underset{\sim}{C}) \qquad \frac{\partial III_C}{\partial \lambda_\alpha^2} = \frac{III_C}{\lambda_\alpha^2}.$$

$$\psi_I = \frac{\partial \psi}{\partial I_C} = \frac{\partial \psi}{\partial \lambda_\alpha^2}\frac{\partial \lambda_\alpha^2}{\partial I_C} = \frac{\partial \psi}{\partial \lambda_\alpha^2}$$

$$\psi_{II} = \frac{\partial \psi}{\partial II_C} = \frac{\partial \psi}{\partial \lambda_\alpha^2}\frac{\partial \lambda_\alpha^2}{\partial II_C} = \frac{\partial \psi}{\partial \lambda_\alpha^2}\left(\frac{1}{2\lambda_\alpha^2}\right)$$

$$\psi_{III} = \frac{\partial \psi}{\partial III_C} = \frac{\partial \psi}{\partial \lambda_\alpha^2}\frac{\partial \lambda_\alpha^2}{\partial III_C} = \frac{\partial \psi}{\partial \lambda_\alpha^2}\left(\frac{\lambda_\alpha^2}{III_C}\right)$$

Substituting these to eq. ①

$$\boxed{\underset{\sim}{S} = \sum_{\alpha=1}^{3} S_{\alpha\alpha} N_\alpha \otimes N_\alpha \qquad S_{\alpha\alpha} = 2\frac{\partial \psi}{\partial \lambda_\alpha^2}} \quad \text{— ②}$$

Push-foward of $\underset{\sim}{S}$ $\longrightarrow$ Cauchy stress (spatial description)

$$\underset{\sim}{\sigma} = J^{-1}\underset{\sim}{F}\,\underset{\sim}{S}\,\underset{\sim}{F}^T = \sum_{\alpha=1}^{3}\frac{2}{J}\frac{\partial \psi}{\partial \lambda_\alpha^2}(\underset{\sim}{F} N_\alpha)\otimes(\underset{\sim}{F} N_\alpha)$$

$$\because \underset{\sim}{F} N_\alpha = \lambda_\alpha \underset{\sim}{n}_\alpha$$

$$\underset{\sim}{\sigma} = \sum_{\alpha=1}^{3}\sigma_{\alpha\alpha}\, n_\alpha \otimes n_\alpha \qquad \sigma_{\alpha\alpha} = \frac{\lambda_\alpha}{J}\frac{\partial \psi}{\partial \lambda_\alpha} = \frac{1}{J}\frac{\partial \psi}{\partial \ln(\lambda_\alpha)}$$

$$\because \frac{2}{J}\frac{\partial \psi}{\partial \lambda_\alpha}\frac{\partial \lambda_\alpha}{\partial \lambda_\alpha^2}(\lambda_\alpha^2)\, n_\alpha \otimes n_\alpha$$

$$= \frac{\lambda_\alpha}{J}\frac{\partial \psi}{\partial \lambda_\alpha}\, n_\alpha \otimes n_\alpha.$$

$$\frac{\partial \ln \lambda_\alpha}{\partial \lambda_\alpha} = \frac{1}{\lambda_\alpha}$$

$$= \frac{1}{J}\frac{\partial \psi}{\partial \ln(\lambda_\alpha)}$$