but $\underset{\sim}{C_e^*} = \underset{\sim}{F_e^{*T}} \underset{\sim}{F_e^*} = \underset{\sim}{Q} \underset{\sim}{F_e^T} \underset{\sim}{F_e} \underset{\sim}{Q^T} = \underset{\sim}{Q} \underset{\sim}{C_e} \underset{\sim}{Q^T}$ (27)

↳ depends on $\underset{\sim}{Q}$

$\underset{\sim}{b_e^*} = \underset{\sim}{F_e^*} \underset{\sim}{F_e^{*T}} = \underset{\sim}{F_e} \underset{\sim}{Q^T} \underset{\sim}{Q} \underset{\sim}{F_e^T} = \underset{\sim}{F_e} \underset{\sim}{F_e^T} = b_e$

therefore $\psi$ and $\tau$, that are fctn of $\underset{\sim}{b_e}$ invariants, are also invariant w.r.t. the rotation $\underset{\sim}{Q}$

∴ $C_p$, $b_e$ → appropriate kinematic measure for constitutive equations

- Rate kinematics

recalling $\underset{\sim}{b_e} = \underset{\sim}{F} \underset{\sim}{C_p^{-1}} \underset{\sim}{F^T}$

$\underset{\sim}{C_p}$ and $\underset{\sim}{b_e}$ are path-dependent and must be obtained by integrating a rate form.

(i.e. cannot be obtained directly from the current configuration alone).

$$\dot{\underset{\sim}{b_e}} = \frac{d}{dt} b_e(\underset{\sim}{F}(t), \underset{\sim}{C_p}(t)) = \frac{\partial \underset{\sim}{b_e}}{\partial \underset{\sim}{F}} : \dot{\underset{\sim}{F}} + \frac{\partial \underset{\sim}{b_e}}{\partial \underset{\sim}{C_p}} : \dot{\underset{\sim}{C_p}}$$

$$= \underbrace{\frac{d \underset{\sim}{b_e}}{dt}\bigg|_{\underset{\sim}{C_p}=\text{const}}}_{\substack{\text{no further permanent deformation} \\ \text{(i.e. trial state of stress)}}} + \underbrace{\frac{d \underset{\sim}{b_e}}{dt}\bigg|_{\underset{\sim}{F}=\text{const}}}_{\substack{\text{results from a change in permanent deformation} \\ \text{(i.e. return mapping)}}}$$

Employing the concept of work conjugacy

(to develop a flow rule)

$$\underbrace{\dot{w}}_{\text{total work rate / unit volume}} = \dot{w}_e + \dot{w}_p$$

in terms of the Kirchhoff stress $\boldsymbol{\tau}$ and velocity gradient $\boldsymbol{L}$.

$$\dot{w} = \boldsymbol{\tau} : \boldsymbol{L} = \boldsymbol{\tau} : \left( \frac{1}{2} \frac{d\boldsymbol{b}_e}{dt}\bigg|_{\boldsymbol{C}_p = const} \cdot \boldsymbol{b}_e^{-1} \right)$$

proof) noting the commutative property of

see example 7.1. (BW Book) $\boldsymbol{b}_e^{-1}\boldsymbol{\tau} = \boldsymbol{\tau}\boldsymbol{b}_e^{-1}$ $\quad \boldsymbol{F}_e^{-T}\boldsymbol{F}_e^{-1}\boldsymbol{\tau} = \boldsymbol{\tau}\boldsymbol{F}_e^{-T}\boldsymbol{F}_e^{-1}$

$$\dot{w} = \frac{1}{2}\mathrm{tr}\left[\boldsymbol{\tau}\left(\dot{\boldsymbol{F}}\boldsymbol{C}_p^{-1}\boldsymbol{F}^T + \boldsymbol{F}\boldsymbol{C}_p^{-1}\dot{\boldsymbol{F}}^T\right)\boldsymbol{F}_e^{-T}\boldsymbol{F}_e^{-1}\right]$$

$$= \frac{1}{2}\mathrm{tr}\left[\boldsymbol{\tau}\dot{\boldsymbol{F}}\boldsymbol{F}_p^{-1}\boldsymbol{F}_p^{-T}\boldsymbol{F}_p^{T}\boldsymbol{F}_e^{T}\boldsymbol{F}_e^{-T}\boldsymbol{F}_e^{-1} + \boldsymbol{\tau}\boldsymbol{F}\boldsymbol{F}_p^{-1}\boldsymbol{F}_p^{-T}\dot{\boldsymbol{F}}^{T}\boldsymbol{F}_e^{-T}\boldsymbol{F}_e^{-1}\right]$$

$$= \frac{1}{2}\mathrm{tr}\left[\boldsymbol{\tau}\underbrace{\dot{\boldsymbol{F}}\boldsymbol{F}^{-1}}_{\boldsymbol{L}}\right] + \frac{1}{2}\mathrm{tr}\left[\boldsymbol{\tau}\boldsymbol{F}_e\boldsymbol{F}_p\boldsymbol{F}_p^{-1}\boldsymbol{F}_p^{-T}\dot{\boldsymbol{F}}^{T}\boldsymbol{F}_e^{-T}\boldsymbol{F}_e^{-1}\right]$$

$$= \frac{1}{2}\boldsymbol{\tau}:\boldsymbol{L} + \frac{1}{2}\mathrm{tr}\left(\boldsymbol{F}_e^{-T}\boldsymbol{F}_e^{-1}\boldsymbol{\tau}\boldsymbol{F}_e\boldsymbol{F}_p^{-T}\dot{\boldsymbol{F}}^{T}\right)$$

$$= \frac{1}{2}\boldsymbol{\tau}:\boldsymbol{L} + \frac{1}{2}\mathrm{tr}\left(\boldsymbol{\tau}\boldsymbol{F}_e^{-T}\boldsymbol{F}_e^{-1}\boldsymbol{F}_e\boldsymbol{F}_p^{-T}\dot{\boldsymbol{F}}^{T}\right)$$

$$= \frac{1}{2}\boldsymbol{\tau}:\boldsymbol{L} + \frac{1}{2}\mathrm{tr}\left(\boldsymbol{\tau}\boldsymbol{F}^{-T}\dot{\boldsymbol{F}}^{T}\right)$$

$$= \frac{1}{2}\boldsymbol{\tau}:\boldsymbol{L} + \frac{1}{2}\boldsymbol{\tau}:\boldsymbol{L} = \boldsymbol{\tau}:\boldsymbol{L}.$$

$$\dot{w}_e = \boldsymbol{\tau} : \boldsymbol{L}_e = \boldsymbol{\tau} : \left( \frac{1}{2} \frac{d\boldsymbol{b}_e}{dt} \boldsymbol{b}_e^{-1} \right) \longrightarrow \text{proof in example 7.2.}$$

$$\boldsymbol{L} = \dot{\boldsymbol{F}} \boldsymbol{F}^{-1} \longrightarrow \boldsymbol{L}_e = \dot{\boldsymbol{F}}_e \boldsymbol{F}_e^{-1}$$

$$\dot{w}_p = \dot{w} - \dot{w}_e$$

$$= \boldsymbol{\tau} : \left( \frac{1}{2} \frac{d\boldsymbol{b}_e}{dt}\bigg|_{C_p=\text{const}} \boldsymbol{b}_e^{-1} \right) - \boldsymbol{\tau} : \left( \frac{1}{2} \frac{d\boldsymbol{b}_e}{dt} \boldsymbol{b}_e^{-1} \right)$$

$$= \boldsymbol{\tau} : \left( \frac{1}{2} \left( \frac{d\boldsymbol{b}_e}{dt}\bigg|_{C_p=\text{const}} - \frac{d\boldsymbol{b}_e}{dt} \right) \boldsymbol{b}_e^{-1} \right)$$

$$= \boldsymbol{\tau} : \left( -\frac{1}{2} \frac{d\boldsymbol{b}_e}{dt}\bigg|_{F=\text{const}} \boldsymbol{b}_e^{-1} \right) = \boldsymbol{\tau} : \boldsymbol{L}_p$$

$$\boldsymbol{L}_p \equiv -\frac{1}{2} \frac{d\boldsymbol{b}_e}{dt}\bigg|_{F=\text{const}} \boldsymbol{b}_e^{-1} \quad \text{(plastic rate-of-deformation)}$$

$$= \frac{1}{2} \boldsymbol{F} \boldsymbol{C}_p^{-1} \dot{\boldsymbol{C}}_p \boldsymbol{F}^{-1} \longrightarrow \text{proof in example 7.3.}$$

recalling $\boldsymbol{b}_e = \boldsymbol{F} \boldsymbol{C}_p^{-1} \boldsymbol{F}^T$ as $C_p$ is path-dependent

$\boldsymbol{b}_e$ is also implicitly path-dependent.

If $\boldsymbol{b}_e$ is determined by time integration of rate-form of $\boldsymbol{b}_e$

then stress can be evaluated by

$$\boldsymbol{\tau} = J \boldsymbol{\sigma} = 2\psi_{I} \boldsymbol{b}_e + 4\psi_{II} \boldsymbol{b}_e^2 + 2 III_{b_e} \psi_{III} \boldsymbol{I}$$

## ⓒ Rate-independent Plasticity.

How to evaluate the plastic rate-of-deformation $\boldsymbol{L}_p$ ?

$\Longrightarrow$ By flow rule

we assume Von Mises plasticity with linear isotropic hardening.

$$f(\boldsymbol{\tau}, \bar{\varepsilon}_p) = \underbrace{\sqrt{\tfrac{3}{2}(\boldsymbol{\tau}' : \boldsymbol{\tau}')}}_{\text{Von Mises equivalent stress}} - \bar{\tau}_y \leq 0$$

where $\bar{\tau}_y = \bar{\tau}_y^{\,0} + H \bar{\varepsilon}_p$

$H \rightarrow$ const. plastic modulus

$\bar{\varepsilon}_p \rightarrow$ hardening variable

$$\boldsymbol{\tau}' = \boldsymbol{\tau} - pJ\boldsymbol{I} \quad \leftarrow \text{from } \boldsymbol{\tau} = J\boldsymbol{\sigma}.$$

if $f(\boldsymbol{\tau}, \bar{\varepsilon}_p) < 0 \rightarrow$ elastically behaves.

if $f(\boldsymbol{\tau}, \bar{\varepsilon}_p) = 0 \rightarrow$ elasto-plastic behavior.

flow rule is defined as

$$\boldsymbol{L}_p = -\frac{1}{2} \frac{d\boldsymbol{b}_e}{dt}\bigg|_{\boldsymbol{F}=\text{const}} \boldsymbol{b}_e^{-1} = \dot{\gamma} \frac{\partial f(\boldsymbol{\tau}, \bar{\varepsilon}_p)}{\partial \boldsymbol{\tau}}.$$

$\dot{\gamma} \rightarrow$ plastic multiplier.

Note that the direction of the plastic strain rate coincides with the gradient of the yield surface.

$\Longrightarrow$ associative flow rule

because of maximum dissipation of plastic work

$\underset{\sim}{\tau} : \underset{\sim}{L} = \underset{\sim}{\tau} : \underset{\sim}{L}e + \underset{\sim}{\tau} : \underset{\sim}{L}p$

maximizing the $\underset{\sim}{\tau} : \underset{\sim}{L}p$ subjected to the constraints $f(\underset{\sim}{\tau}, \bar{\varepsilon}_p) \leq 0$ leads the eqn.

$$\underset{\sim}{L}p = \dot{\gamma} \frac{\partial f(\underset{\sim}{\tau}, \bar{\varepsilon}_p)}{\partial \underset{\sim}{\tau}}$$

Substituting $f$ to this eqn.

$$f(\tau, \bar{\varepsilon}_p) = \left[ \frac{3}{2}(\tau_{ij} - pJ\delta_{ij})(\tau_{ij} - pJ\delta_{ij}) \right]^{1/2} - \bar{\tau}_y$$

$$= \left[ \frac{3}{2}(\tau_{ij}\tau_{ij} - 2pJ\tau_{ii} + 3p^2J^2) \right]^{1/2} - \bar{\tau}_y$$

$$= \left[ \frac{3}{2}\tau_{ij}\tau_{ij} - 3pJ\tau_{ii} + \frac{9}{2}p^2J^2 \right]^{1/2} - \bar{\tau}_y$$

$$\underset{\sim}{L}p = \dot{\gamma}\frac{\partial f}{\partial \tau_{kl}} = \dot{\gamma}\,\frac{1}{2}\left[ \frac{3}{2}\tau_{ij}\tau_{ij} - 3pJ\tau_{ii} + \frac{9}{2}p^2J^2 \right]^{-1/2}$$

$$\left( \frac{3}{2}\delta_{ik}\delta_{jl}\tau_{ij} + \frac{3}{2}\tau_{ij}\delta_{ik}\delta_{jl} - 3pJ\delta_{ik}\delta_{il} \right)$$

$$= \dot{\gamma} \frac{1/2}{\sqrt{\frac{3}{2}(\underset{\sim}{\tau}' : \underset{\sim}{\tau}')}} [\underbrace{3\tau_{kl} - 3pJ\underset{\sim}{I}}_{= 3\underset{\sim}{\tau}'}]$$

$$= \dot{\gamma} \frac{\underset{\sim}{\tau}'}{\sqrt{\frac{2}{3}(\underset{\sim}{\tau}' : \underset{\sim}{\tau}')}}$$

following the work-hardening approach.

$$\dot{\omega}_p = \bar{\tau}\dot{\bar{\varepsilon}}_p = \underset{\sim}{\tau} : \underset{\sim}{L}_p = \underset{\sim}{\tau} : \dot{\gamma}\frac{\underset{\sim}{\tau}'}{\sqrt{\frac{2}{3}(\underset{\sim}{\tau}':\underset{\sim}{\tau}')}}$$

$$= \sqrt{\frac{3}{2}(\underset{\sim}{\tau}':\underset{\sim}{\tau}')}\,\dot{\gamma} = \dot{\gamma}\bar{\tau} \Longrightarrow \boxed{\dot{\gamma} = \dot{\bar{\varepsilon}}_p}$$

Note) $\underset{\sim}{\tau} : \underset{\sim}{\tau}' = (\underset{\sim}{\tau}' + pJ\underset{\sim}{I}) : \underset{\sim}{\tau}'$

$$= \underset{\sim}{\tau}' : \underset{\sim}{\tau}' + pJ\,\cancel{tr(\underset{\sim}{\tau}')}^{=0.}$$

Remark) Volumetric component of $\underset{\sim}{L}_p$, that is, $tr(\underset{\sim}{L}_p)$ is zero!

$$tr(\underset{\sim}{L}_p) = \dot{\gamma}\frac{tr(\underset{\sim}{\tau}')}{\sqrt{\frac{2}{3}(\underset{\sim}{\tau}':\underset{\sim}{\tau}')}} = 0. \Longrightarrow tr(\underset{\sim}{\tau}') = 0$$

$\det(\underset{\sim}{F}_p) = 1 \rightarrow$ plastic deformation is incompressible.

⊙ Principal Directions.