

# Lecture 1: Free Vibration of Damped Single Degree of Freedom (SDOF) System

### Outline



- 1. Learning Objectives
- 2. Free Vibration of Undamped SDOF System
- 3. Free Vibration of Damped SDOF System
- 4. Experimental Determination of Natural Frequency and Damping Factor
- 5. Problem Statements
- 6. Test Setup and Equipment
- 7. Users Manual of NI-LabVIEW VI Program
- 8. Experimental Test Results
- 9. Project
- 10. Reference



# **Learning Objectives**



- Objective 1: Analyze dynamic response of a Single Degree of Freedom (SDOF) system under free vibration
- Objective 2: Determine the stiffness of one-story building structure using experimentally measured acceleration data
- Objective 3: Determine the natural frequency and damping factor using the logarithmic decrement method through experimental testing



# 2 Free Vibration of Undamped SDOF System



#### • Considering a mass-spring system





Using Newton's Second Law,

$$\sum F_x = m\ddot{u} = -f_s + p(t)$$

 $m\ddot{u} + ku = p(t)$ 

Equation of Motion

The solution is based on the initial conditions

 $u(0) = u_0 \quad \text{and} \quad \dot{u}(0) = v_0$ 



#### Definitions

• Circular Natural Frequency (rad/sec)  $\rightarrow \omega_n = \sqrt{\frac{k}{m}} = 2\pi f_n$ 

□ Period (sec) → 
$$T_n = \frac{2\pi}{\omega_n}$$
  $T_n = \frac{1}{f_n}$ 

• Free vibration, that is, p(t)=0. Then the E.O.M is

 $m\ddot{u} + ku = 0$ 

- □ Its general solution take a form of  $u(t) = A_1 \cos \omega_n t + A_2 \sin \omega_n t$
- A<sub>1</sub> and A<sub>2</sub> have values dependent on the initial conditions  $u_0$  and  $v_0$   $\ddot{u} = -A_1 \omega_n^2 \cos \omega_n t - A_2 \omega_n^2 \sin \omega_n t$   $m(-A_1 \omega_n^2 \cos \omega_n t - A_2 \omega_n^2 \sin \omega_n t) + k(A_1 \cos \omega_n t + A_2 \sin \omega_n t) = 0$   $A_1(-m\omega_n^2 + k)\cos \omega_n t + A_2(-m\omega_n^2 + k)\cos \omega_n t = 0$ = 0  $\omega_n^2 = \frac{k}{m}$   $\omega_n = \sqrt{\frac{k}{m}}$



#### **To find A\_1 and A\_2**

$$u(0) = u_0 = A_1 \cos(0) + A_2 \sin(0)$$
$$A_1 = u_0$$

$$u(t) = u_0 \cos \omega_n t + \frac{v_0}{\omega_n} \sin \omega_n t$$

This can also be written



$$\dot{u}(0) = v_0 = -A_1 \omega_n \sin(0) + A_2 \omega_n \cos(0)$$

$$= A_2 \omega_n \qquad A_2 = v_0 / \omega_n$$

$$u_0 \quad \text{and} \quad v_0 / \omega_n$$
Two rotating
Vectors off by 90°
$$\omega_n t \qquad U$$

$$w_0 / \omega_n$$

$$U = \sqrt{u_0^2 + \left(\frac{v_0}{\omega_n}\right)^2} \qquad \alpha = \tan^{-1} \frac{v_0 / \omega_n}{u_0}$$



# **B** Free Vibration of Damped SDOF System



→ u(t)

m



$$m\ddot{u} + c\dot{u} + ku = p(t)$$
$$\ddot{u} + 2\zeta\omega_n\dot{u} + \omega_n^2u = \frac{p(t)}{m}$$

Definition:



where  $c_{cr} = 2m\omega_n = 2\sqrt{mk}$  is the critical damping

For free vibration, p(t)=0 and given initial conditions,  $u_0 = u(0)$ ,  $v_0 = \dot{u}(0)$ 

 $u(t) = Ce^{st}$  $\dot{u}(t) = Cse^{st}$  $\ddot{u}(t) = Cs^{2}e^{st}$ 

Substituting to the Equation of Motion



$$Ce^{st} \left( s^{2} + 2\zeta \omega_{n} s + \omega_{n}^{2} \right) = 0$$
  
=0, Characteristic Equation(C.E.)  
Then the roots of the C.E. are  
$$\sqrt{-1}$$
$$s_{1,2} = -\zeta \omega_{n} \pm \sqrt{(\zeta \omega_{n})^{2} - \omega_{n}^{2}} = -\zeta \omega_{n} \pm i\omega_{n} \sqrt{1 - \zeta^{2}}$$
$$\omega_{d} = \omega_{n} \sqrt{1 - \zeta^{2}} \rightarrow \text{Damped natural frequency}$$

I) Undamped case (ζ=0)

$$s_1 = +i\omega_n$$
  $s_2 = -i\omega_n$ 

**2)** Underdamped case ( $0 < \zeta < 1$ )

$$s_1 = -\zeta \omega_n + i\omega_d$$
$$s_2 = -\zeta \omega_n - i\omega_d$$



**3**) Critically damped ( $\zeta$ =1)

 $S_{1,2} = -\omega_n$ 

4) Overdamped (ζ>1)

$$s_{1,2} = -\zeta \omega_n \pm \omega_n \sqrt{\zeta^2 - 1}$$
  $\rightarrow$  Two negative real roots





#### • For underdamped system ( $\zeta$ <1)

$$\begin{split} s_{1,2} &= -\zeta \omega_n \pm i \omega_d \\ u(t) &= C e^{st} = C_1 e^{s_1 t} + C_2 e^{s_2 t} \\ &= C_1 e^{(-\zeta \omega_n + i \omega_d)t} + C_2 e^{(-\zeta \omega_n - i \omega_d)t} \\ &= e^{-\zeta \omega_n t} \left( C_1 e^{i \omega_d t} + C_2 e^{-i \omega_d t} \right) = e^{-\zeta \omega_n t} \left( A_1 \cos \omega_d t + A_2 \sin \omega_d t \right) \\ \dot{u}(t) &= -\zeta \omega_n e^{-\zeta \omega_n t} A_1 \cos \omega_d t - e^{-\zeta \omega_n t} A_1 \omega_d \sin \omega_d t \\ &\quad -\zeta \omega_n e^{-\zeta \omega_n t} A_2 \sin \omega_d t + e^{-\zeta \omega_n t} A_2 \omega_d \cos \omega_d t \end{split}$$

Determining  $A_1$  and  $A_2$  by applying the initial conditions,  $u(0) = u_0 = A_1$ 

$$\dot{u}(0) = v_0 = -\zeta \omega_n A_1 + A_2 \omega_d \qquad A_2 = \frac{v_0 + \zeta \omega_n u_0}{\omega_d}$$



$$u(t) = e^{-\zeta \omega_n t} \left\{ u_0 \cos \omega_d t + \left( \frac{v_0 + \zeta \omega_n u_0}{\omega_d} \right) \sin \omega_d t \right\}$$

Using the rotating vector technique,

$$u(t) = U e^{-\zeta \omega_n t} \cos \omega_d \left( t - \frac{\alpha}{\omega_d} \right)$$

$$U = \sqrt{u_0^2 + \left(\frac{v_0 + \zeta \omega_n u_0}{\omega_d}\right)^2}$$

$$\alpha = \tan^{-1}\left(\frac{v_0 + \zeta \omega_n u_0}{u_0 \omega_d}\right)$$



- For critically damped system ( $\zeta$ =1)
  - Solutions of characteristic equation

$$s = -\zeta \omega_n = -\omega_n \quad \Rightarrow \text{ Only one real solution}$$
$$u(t) = (C_1 + C_2 t)e^{-\zeta \omega_n t}$$
$$\dot{u}(t) = -\zeta \omega_n C_1 e^{-\zeta \omega_n t} + C_2 e^{-\zeta \omega_n t} - \zeta \omega_n C_2 t e^{-\zeta \omega_n t}$$
$$u(0) = u_0 = C_1$$
$$\dot{u}(0) = v_0 = -\zeta \omega_n C_1 + C_2$$
$$C_2 = v_0 + \zeta \omega_n u_0$$

$$u(t) = \left[u_0 + \left(v_0 + \zeta \omega_n u_0\right)t\right]e^{-\omega_n t}$$



- For overdamped system ( $\zeta$ >1)
  - Solutions of characteristic equation

$$s_{1,2} = -\zeta \omega_n \pm \omega_n \sqrt{\zeta^2 - 1}$$

• Letting  $\omega^* = \omega_n \sqrt{\zeta^2 - 1}$ 

$$u(t) = e^{-\zeta \omega_n t} \left( C_1 \cosh \omega^* t + C_2 \sinh \omega^* t \right)$$

Applying the initial conditions

$$u(t) = e^{-\zeta \omega_n t} \left[ u_0 \cosh \omega^* t + \frac{v_0 + \zeta \omega_n u_0}{\omega^*} \sinh \omega^* t \right]$$





In complex plan (s-plane),





□ Find the stiffness of the table.





- Observe video of vibrating one-story building structure through remote collaboratory system.
- Count the time it takes until the amplitude is completely damped out.
- Is this structure underdamped, overdamped, critically damped or undamped?



Consider a wind farm. P=400 lb force displaces the wind turbine mass by 0.5 in. The mass is released at the point. At the end of two



complete cycles, the displacement was 0.2 in and the time is 2 sec.

Determine (a) the undamped natural frequency, (b) the effective stiffness, (c) the effective mass and (d) the effective damping factor.



# 4 Experimental Determination of Natural Frequency and Damping Factor



#### Static-Displacement Method

• One approach to measure  $\omega_n$  is to measure the static displacement of a mass when the weight is applied.



$$W = mg = ku_{st}$$
$$\omega_n^2 = \frac{k}{m} = \frac{g}{u_{st}}$$

Note: m and k are not needed.

Free Vibration Method

□ For most of structural system ( $\zeta$ <0.2),  $\omega_d \approx \omega_n$  is a valid approximation.



If 1.5 cycles are executed in 0.6 sec, then

$$f_n = \frac{1.5 \text{ cycles}}{0.6 \text{ sec}} = 2.5 \text{ Hz}$$



- Logarithmic Decrement Method
  - Under free vibration, the displacement response is



□ Similarly,

$$\zeta = \frac{1}{2\pi} \ln \left( \frac{\dot{u}(t)}{\dot{u}(t+T_d)} \right) = \frac{1}{2\pi} \ln \left( \frac{\ddot{u}(t)}{\ddot{u}(t+T_d)} \right)$$

$$\frac{u(t)}{u(t+T_d)} = \frac{e^{-\zeta \omega_n t}}{e^{-\zeta \omega_n (t+T_d)}} = e^{\zeta \omega_n T_d} = e^{\frac{2\pi\zeta}{\sqrt{1-\zeta^2}}}$$
$$\left( \begin{array}{c} \operatorname{Rec all} & T_d = \frac{2\pi}{\omega_d} = \frac{2\pi}{\sqrt{1-\zeta^2}} \end{array} \right)$$
$$\ln\left(\frac{u(t)}{u(t+T_d)}\right) = \frac{2\pi\zeta}{\sqrt{1-\zeta^2}} = \delta$$
For  $\zeta < 0.2$   $\delta = 2\pi\zeta$ 
$$\int \zeta = \frac{1}{2\pi} \ln\left(\frac{u(t)}{u(t+T_d)}\right)$$

- This equation shows free vibration response of underdamped SDOF system.
- Assuming that we are about to run a lab test for verification of this closed-form solution of the mass-dashpot-spring model, please identify parameters that need to be measured and parameters that need to be computed.

$$u(t) = e^{-\zeta \omega_n t} \left\{ u_0 \cos \omega_d t + \left( \frac{v_0 + \zeta \omega_n u_0}{\omega_d} \right) \sin \omega_d t \right\}$$

Parameters from experiment: \_\_\_\_\_

Parameters from calculation:





# Problem Statements



In this test, an one-story shear building structure will be used in order to measure acceleration under free vibration and determine its effective mass value, the natural frequency, and the damping factor using the logarithmic decrement method. Free Vibration



After applying an initial displacement  $(u_0)$  on the floor, release the structure in order to make the structure freely vibrate. Using an accelerometer connected to an AnyLoggerS-V/ICP, measure the acceleration signals through a NI-LabVIEW VI program. Calculate the stiffness of the structure. And determine the damped natural frequency and the damping factor. Compare with results from the NI-LabVIEW VI program.

### **Procedures**



- 1) You are provided with the measured dimensions of this test structure as well as the calibration information for the sensors. The mass of floor of the structure (including the accelerometer and mounting plate) is 1550.0 gram. Also, sensor readings are given in gravities (1g=9.8 m/sec<sup>2</sup>). The saturation level (maximum reading) for the two accelerometers is +/-50g. Note that in some cases there may be an offset in the accelerometer. However, the saved acceleration data has no DC-offset because the NI-LabVIEW program automatically removes the DC-offset before saving.
- 2) Set the number of samples to be recorded, the sampling rate, the filter type/order, the computed stiffness on the NI front panel.
   (Please see the users manual of the provided NI-LabVIEW VI program for detail procedures related to the VI program)
- 3) Impose the predetermined initial displacement and make a free vibration of the test structure.
- 4) During the free vibration, immediately click "start" bottom of the NI-LabVIEW VI program and record data for more than preset recording time (=the number of samples/sampling rate). For details, please see "Users manual of NI-LabVIEW VI programs"



- 5) Observe the free vibration response. Describe the free vibration response from the beginning to the end of the test.
- 6) After the lab test, use the saved five files: (a) measured accelerations; (b) velocities and displacements which are integrated from the measured accelerations; (c) computed displacements; (d) computed velocities, and (e) computed accelerations, in order to compute the effective mass, the natural frequency, and the damping factor.
- 7) Compare the calculated results with those from NI-LabVIEW VI program provided.
- 8) After the lab test, formulate a mathematical model of the test structure. Use a simple mass-spring-damper system. Derive the freevibration response following the lecture notes. Using MATLAB, make plots of the free vibration responses from the data (the natural frequency, the damping factor and the stiffness) you computed from the experiments. Show the responses from the experiment and the mathematical model on the same plot.

#### Reports



- Write a full report using the instructions provided in class. Organize your report into sections (e.g. Introduction, Procedures, Results, Discussion, Summary, References). Write concisely and clearly.
- Include the following: (1) A schematic diagram and description of the test equipment. (2) Plots of the free vibration responses recorded by NI-LabVIEW VI program during the experiment. (3) Estimated the stiffness, the natural frequency, damping factor from the experimental data (4) Free-body diagrams, and the equation of motion for your mathematical model. (5) Representative plots of the free vibration response of the dynamic system. and (6) Assumptions and possible explanations for differences between theory and observation.



# Test Setup and Equipment

#### **Test Setup**





#### **Test Structure**





Total weight :1550.0 g (including sensor)

Unit of length : mm

#### **Test Structure- Drawing**







- NI-PXI 8105 Controller
- □ ICP type Dytran triaxial accelerometer (3093B1)
- One-channel AnyLogger (Korea Maintenance Co., LTD): AnyLoggerS-V/ICP.
- □ NI-LabVIEW 8.6

### **NI-PXI 8105 Controller**





#### **Specification**

- Intel Core Duo Processor T2500(2.0 GHz dual core)
- 512 MB (1 x 512 MB DIMM) dual channel 667 MHz DDR2 RAM standard,4 GB (2 x 2 GB DIMMs) maximum
- □ Integrated I/O
  - □ 10/100/1000BASE-TX Ethernet
  - 4 Hi-Speed USB ports
  - ExpressCard/34 slot
  - DVI-I video connector
  - GPIB (IEEE 488) controller
  - RS232 serial port
  - □ IEEE 1284 ECP/EPP parallel port
  - Integrated hard drive
- Internal PXI trigger bus routing
- Watchdog timer Software
- □ Hard drive-based recovery image PXI System





Model 3093B1 Dytran Triaxial Accelerometer

Specification	Value	Uint
Weight	13.5	Grams
Size(Height x Width x Depth)	0.54 x0.59 x 0.59	Inch
Sensitivity	100	mV/G
Ranges	+/-50	G
Frequency Response	2 to 5000	Hz
Equivalent Electrical Noise	0.007	G, RMS
Linearity	1	% F.S.
Temp. Range	-60 ~ +250	°F
Supply Current Range each axis	2 to 20	mA
Supply Voltage Range each axis	+18 to +30	VDC
Output impedance	100	OHMS





#### AnyloggerS-V/ICP for acceleration transmitter AnyLoggerS-B for strain transmitter



#### Specification

Contents	AnyloggerS-V/ICP	AnyLoggerS-B
Support Num. of Channel	1	1
Input Voltage Range	-5 ~ 5V	0 ~ 3V
Gain	1,2,5,10,20,50,100	50,100,250,500,1000,2500,5000
Programmable	10~1000Hz (10,20,50,100,200,500,1000)	10 ~ 1000Hz
Lower Pass Filter		10,20,50,100,200,500,1000
Prog. Offset	0 ~ 5V(12Bit)	0 ~ 3.3V(12Bit)
Max Sampling Rate	1000Hz	1000Hz
Exciting Voltage	24V(Only ICP Type)	3.3V±0.5%
Connector	BNC Connector	4Pin Circular Connector
SIM Usage	No	Yes
Power Consumption (w/o sensor)	150mA	100mA(w/o SIM)
Internal Battery	Li-ion Rechargable 1500mAh x 2EA(Serial)	Li-ion Rechargable1500mAh x 2EA(Serial)
<b>Ext. Power Requirement</b>	5V	5V
<b>Operation Temperature</b>	-10 ~ 80°C	-10 ~ 80°C
Sync. Accuracy	< 10ms	< 10ms
Dimension	800 x 973 x 353	800 x 883 x 353
Weight	210g	210g
ADC Resolution	Differential 16Bit	Differential 16Bit
Measurement Accuracy	F.S. 0.1%	F.S. 0.1%
Sensor Connectibility	Voltage or ICP source	Bridge type sensor
Signal Ripple	Depends on Gain	Depends on Gain
Communication	Bluetooth v1.2 class1 18dBm(w/o ant.)	Bluetooth v1.2 class1 18dBm(w/o ant.)
Radio Frequency Range	2.402 ~ 2.480GHz	2.402 ~ 2.480GHz
Transmission Method	FHSS(freq. Hopping Spread Spectrum)	FHSS
Modulation Method	GFSK(Gaussian-filtered Freq. Shift Keying)	GFSK
Approvals	MIC, FCC, CE	MIC, FCC, CE



□ LabVIEW (Laboratory Virtual Instrument Engineering Workbench) is a graphical programming language that uses icons instead of lines of text to create applications.

□ In contrast to text-based programming languages, where instructions determine the order of program execution, LabVIEW uses dataflow programming, where the flow of data through the nodes on the block diagram determines the execution order of the VIs and functions. VIs, or virtual instrument, are LabVIEW programs that imitate physical instruments.





# **Users Manual of NI-LabVIEW VI** Program

# Users Manual of LabVIEW VI Prog. (1)



- 1) Set Parameter to "Monitoring Start" and click "Set"
- 2) Set Path Create empty files in "Path" where the original raw acceleration data will be saved and in "Dummy path" where dummy data will be saved
- 3) Turn on "Save Mode" Save the original raw data to the file
- 4) Set files- Create files where measured and computed data will be saved and link them to this panel by choosing them.
- 5) Sampling Rate Default 1KHz
- 6) Number of data Input the number of data for display and analysis
- 7) Set Filter Type Choose "Bandpass"
- 8) Set Filter Order
- 9) Set Low Cutoff Freq
- 10) Set High Cutoff Freq
- **11) Run** Run the front panel while the structure is vibrating
- 12) Stop After more than specified times (# of Data/ Samp. Rate), click "Stop". Users can control the monitoring time.



# Users Manual of LabVIEW VI Prog. (2)



After a while, measured acceleration, integrated velocity and displacement data will be plotted. **Damping factor** and **natural frequency** will appear. To plot computed free vibration response, follow steps below.

#### 12) Turn off "Save Mode"

- 13) Copy Data copy and past measured initial disp. And velocity to text box below. Note that users have to compute undamped natural frequency.
- 14) Run- Run the front panel. Wait until computed responses are plotted. It may takes several minutes depending on the system.

Compute the undamped natural frequency and enter the value here



### Users Manual of LabVIEW VI Prog. (3)



Finally, users can compare the measured response with the computed response.

Natural frequency and damping factor are automatically obtained.



Users Manual of LabVIEW VI Prog. (4) Km 한국유지관리(주)

- □ Two-stack sequences were used
- In the first frame, users should define correct **IP address** and **port number** based on the equipments following the Anylogger manual.



#### First Stack of Block Diagram of VI Program

Users Manual of LabVIEW VI Prog. (5) Km 한국유지관리(주)

- In the second frame, the saved original raw data is read from the path for calibration and final saving
- Wavelet denoising filter is used to remove noise for generating the converted data
- The acc. data is converted to velocity and disp. data using integration
- Bandpass filter is used to pass only the targeted frequency.
- From the disp. data, initial disp. and velocity are automatically obtained and displayed on the front panel.
- Based on the initial disp. and velocity, the analytical response of disp., velocity and acc. are plotted on the panel

#### dt=1/sampling rate Save Test Acc Wavelet Denoise2 Gain $\sqrt{\sigma\mu}$ ~~~ 61 Path 0.0001525878906 MEAN Test\_acc Ísruat โลเบสเ Read From Measurement X=disí 0.0973 File Sensitivity for the MEAN Signals 1000 # of Data Ini. Disp (m) Ini. Vel. 1.230 1.230 K=1; H cutoff frea sz=size(acc,2); MEAN L dutoff fred for i=2000:sz-1 for i=1:size if(K>4); 1000 Filter order t = sRat break; , ... DIS(i)=e di end VEL(i)= Samp. Rate if (acc(i)<0 & ACC(i)= x=abs(acci 1.23 . ..... z=x-y; Displacement if z<0; Frequency dis=DIS(1,1 ind(K)=i: vel=VEL(1. K=K+1; Save Test Vel. acc=ACC(1) ind(K)=i-Path K=K+1; ₿+₿ ⊞-⊞ 1000 1000 Acceleration Damping Factor 1.23 0 Ini Disp. from test Test\_disp Ini Vel, from test Save Analytical DispSave Analytical Vel. Save Analytical Acc. S Pat Save Test Disp. Path Path

#### Sensitivity of accelerometer (V/g)

Second Stack of Block Diagram of VI Program



# 8 Experimental Test Results

### **Test Results (1)**







 The damping factor and natural frequency were identified as (less than 1%) and \_\_\_\_\_Hz, respectively.

Compute the stiffness of the column that is assumed to be fully restrained at the top and bottom. Note that there are two columns

Compute the effective mass based on the stiffness and measured undamped natural frequency.







- Following the procedures explained in this lecture note, conduct the experiment and write a report.
  - Write a full report using the instructions provided in class.
     Organize your report into sections (e.g. Introduction, Procedures, Results, Discussion, Summary, References). Write concisely and clearly.
  - Include the following: (1) A schematic diagram and description of the test equipment. (2) Plots of the free responses recorded by NI-LabVIEW VI program during the experiment. (3) Estimated the stiffness, the natural frequency, damping ratio from the experimental data (4) Free-body diagrams, and the equation of motion for your mathematical model. (5) Representative plots of the free vibration response of the dynamic system. and (6) Assumptions and possible explanations for differences between theory and observation.







 Roy R. Craig, Jr. and Andrew J. Kurdila, "Fundamentals of Structural Dynamics", 2<sup>nd</sup> Ed. Wiley