



Lecture 2: Time History Analysis of Free Vibration of Damped Single Degree of Freedom System

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Learning Objectives

Learning Objectives

- ❑ Objective 1: Simulate dynamic response of a Single Degree of Freedom (SDOF) system under free vibration using time integration methods: 1) Central Difference Method, and 2) Newmark Methods
 - ❑ Objective 2: Compare simulation results with experimentally measured responses
 - ❑ Objective 3: Understand causes of the observed errors in the time integration methods
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2

Numerical Evaluation of Dynamic Response of SDOF System

Time Stepping Methods (1)

- ❑ It is hard to get analytical solutions for responses of the structures under arbitrary excitations.
- ❑ In this case, numerical solutions are suitable.
- ❑ It should be noted that all numerical integration methods have issues such as
 - ❑ Accuracy
 - ❑ Convergence
 - ❑ Stability
- ❑ Equation of motion is expressed in a form of second order differential equation in structural dynamics

$$m\ddot{u} + c\dot{u} + f_s(u, \dot{u}) = p(t) \quad \text{or} \quad -m\ddot{u}_g(t)$$

$$\text{such that } u_0 = u(0) \quad \dot{u}_0 = \dot{u}(0)$$

Time Stepping Methods (2)

- For time stepping methods, the continuous equation of motion should be discretized in time domain as follows

$$m\ddot{u}_i + c\dot{u}_i + ku_i = p_i$$

$$p(t_i) = p_i \quad i = 0, 1, \dots, N \quad \Delta t_i = t_{i+1} - t_i$$

- And the time stepping method provides solution algorithm to get dynamic responses at each time station

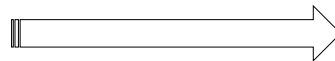
$$m\ddot{u}_i + c\dot{u}_i + ku_i = p_i$$

$$m\ddot{u}_{i+1} + c\dot{u}_{i+1} + ku_{i+1} = p_{i+1}$$

$$u_i = u(i\Delta t)$$

$$\dot{u}_i = \dot{u}(i\Delta t)$$

$$\ddot{u}_i = \ddot{u}(i\Delta t)$$



$$u_{i+1} = u((i+1)\Delta t)$$

$$\dot{u}_{i+1} = \dot{u}((i+1)\Delta t)$$

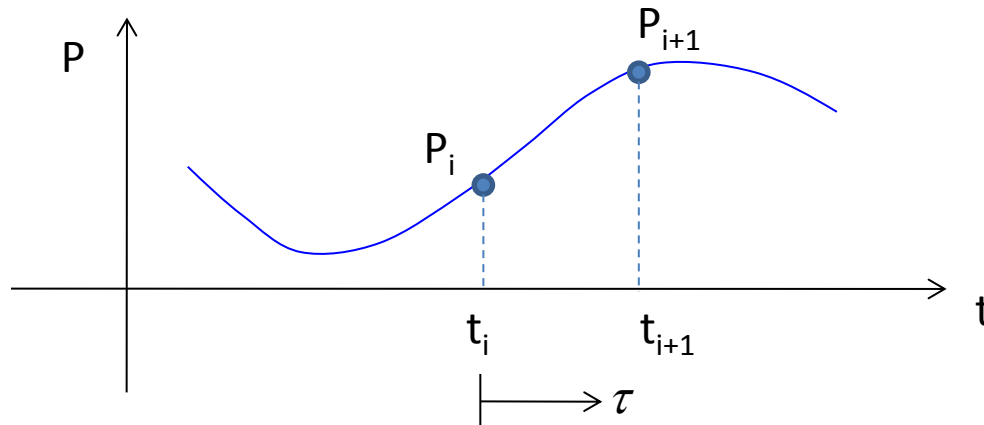
$$\ddot{u}_{i+1} = \ddot{u}((i+1)\Delta t)$$

Time Stepping Methods (3)

- ❑ There are many different methods
 - ❑ Central Difference Method (CDM)
 - ❑ Newmark Methods (i.e. linear & average acceleration methods)
 - ❑ Wilson θ Method
 - ❑ Other time integration methods
 - ❑ Houbolt Method
 - ❑ Hilber-Hughes-Taylor Method
 - ❑ SSpj Method
 - ❑ Symmetric Successive Quadrature Method
 - ❑ Etc.

- ❑ In this lecture note, the first two methods (CDM and Newmark Method) will be presented.

Numerical Solution Based Interpolation of Excitation Function (1)



- Response in between t_i and t_{i+1} can be considered to be the sum of
 - Free vibration due to initial conditions u_i and \dot{u}_i
 - Response due to step force p_i with zero initial conditions

- Response due to ramp force $\left(\frac{\Delta p_i}{\Delta t_i}\right)\tau$ with zero initial conditions

$$u_{i+1} = u_i \cos(\omega_n \Delta t_i) + \frac{\dot{u}_i}{\omega_n} \sin(\omega_n \Delta t_i) + \frac{p_i}{k} (1 - \cos(\omega_n \Delta t_i))$$
$$+ \frac{\Delta p_i}{k} \left(1 - \frac{\sin(\omega_n \Delta t_i)}{\omega_n \Delta t_i} \right)$$

Numerical Solution Based Interpolation of Excitation Function (2)

$$\frac{\dot{u}_{i+1}}{\omega_n} = -u_i \sin(\omega_n \Delta t_i) + \frac{\dot{u}_i}{\omega_n} \cos(\omega_n \Delta t_i) + \frac{p_i}{k} \sin(\omega_n \Delta t_i) + \frac{\Delta p_i}{k} \left(\frac{1}{\omega_n \Delta t_i} - \frac{\cos(\omega_n \Delta t_i)}{\omega_n \Delta t_i} \right)$$

- Then, the recurrence formula can be established after substituting

$$\Delta p_i = p_{i+1} - p_i$$

$$u_{i+1} = Au_i + B\dot{u}_i + Cp_i + Dp_{i+1}$$

$$\dot{u}_{i+1} = A'u_i + B'\dot{u}_i + C'p_i + D'p_{i+1}$$

- If the time step Δt is constant, the coefficients A, B,..., D' need to be computed only once.
- For the coefficients, refer to the following table

Numerical Solution Based Interpolation of Excitation Function (3)

□ Coefficients in Recurrence Formula ($\zeta < 1$)

$$A = e^{-\zeta \omega_n \Delta t} \left(\frac{\zeta}{\sqrt{1 - \zeta^2}} \sin \omega_D \Delta t + \cos \omega_D \Delta t \right)$$

$$B = e^{-\zeta \omega_n \Delta t} \left(\frac{1}{\omega_D} \sin \omega_D \Delta t \right)$$

$$C = \frac{1}{k} \left\{ \frac{2\zeta}{\omega_n \Delta t} + e^{-\zeta \omega_n \Delta t} \left[\left(\frac{1 - 2\zeta^2}{\omega_D \Delta t} - \frac{\zeta}{\sqrt{1 - \zeta^2}} \right) \sin \omega_D \Delta t - \left(1 + \frac{2\zeta}{\omega_n \Delta t} \right) \cos \omega_D \Delta t \right] \right\}$$

$$D = \frac{1}{k} \left[1 - \frac{2\zeta}{\omega_n \Delta t} + e^{-\zeta \omega_n \Delta t} \left(\frac{2\zeta^2 - 1}{\omega_D \Delta t} \sin \omega_D \Delta t + \frac{2\zeta}{\omega_n \Delta t} \cos \omega_D \Delta t \right) \right]$$

$$A' = -e^{-\zeta \omega_n \Delta t} \left(\frac{\omega_n}{\sqrt{1 - \zeta^2}} \sin \omega_D \Delta t \right)$$

$$B' = e^{-\zeta \omega_n \Delta t} \left(\cos \omega_D \Delta t - \frac{\zeta}{\sqrt{1 - \zeta^2}} \sin \omega_D \Delta t \right)$$

$$C' = \frac{1}{k} \left\{ -\frac{1}{\Delta t} + e^{-\zeta \omega_n \Delta t} \left[\left(\frac{\omega_n}{\sqrt{1 - \zeta^2}} + \frac{\zeta}{\Delta t \sqrt{1 - \zeta^2}} \right) \sin \omega_D \Delta t + \frac{1}{\Delta t} \cos \omega_D \Delta t \right] \right\}$$

$$D' = \frac{1}{k \Delta t} \left[1 - e^{-\zeta \omega_n \Delta t} \left(\frac{\zeta}{\sqrt{1 - \zeta^2}} \sin \omega_D \Delta t + \cos \omega_D \Delta t \right) \right]$$

Example

Initial conditions

- Substituting $\omega_n=6.283$, $k=10$ and $\Delta t_i=0.1$ in the equations for A, B, C, ..., D' gives

$$\begin{array}{llll} A=0.8090 & B=0.09355 & C=0.00955 & D=0.00955 \\ A'=-3.6932 & B'=0.809 & C'=0.1847 & D'=0.1847 \end{array}$$

Apply the recurrence equation

- The resulting computations are summarized in the Table.
- Numerical solution of displacement

t_i	p_i	Cp_i	Dp_{i+1}	$B\ddot{u}_i$	\ddot{u}_i	Au_i	u_i	Theoretical u_i
0.0	0.0000	0.0000	0.0477	0.0000	0.0000	0.0000	0.0000	0.0000
0.1	5.0000	0.0477	0.0827	0.0864	0.9233	0.0386	0.0477	0.0333
0.2	8.6602	0.0827	0.0955	0.2894	3.0931	0.2067	0.2554	0.2405
0.3	10.0000	0.0955	0.0827	0.4682	5.0047	0.5454	0.6742	0.6789
0.4	8.6603	0.0827	0.0477	0.4682	5.0047	0.9642	1.1918	1.2312
0.5	5.0000	0.0477	0.0000	0.2030	2.1698	1.2644	1.5628	1.6364
0.6	0.0000	0.0000	0.0000	-0.2894	-3.0931	1.2257	1.5151	1.6031
0.7	0.0000	0.0000	0.0000	-0.7575	-8.0978	0.7575	0.9364	0.9907
0.8	0.0000	0.0000	0.0000	-0.9364	-10.0095	0.0000	0.0000	0.0000
0.9	0.0000	0.0000	0.0000	-0.7575	-8.0979	-0.7575	-0.9364	-0.9907
1.0	0.0000				-3.0931		-1.5151	-1.6031

Example

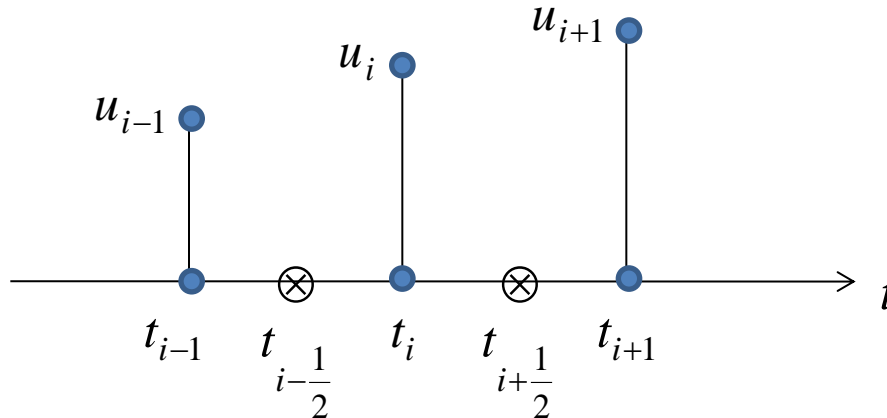
□ Numerical solution of velocity

t_i	p_i	$C'p_i$	$D'p_{i+1}$	$A'u_i$	u_i	$B'\dot{u}_i$	\dot{u}_i	Theoretical \dot{u}_i
0.0	0.0000	0.0000	0.9233	0.0000	0.0000	0.0000	0.0000	0.0000
0.1	5.0000	0.9233	1.5992	- 0.1763	0.0477	0.7470	0.9233	0.9769
0.2	8.6602	1.5992	1.8466	- 0.9434	0.2554	2.5024	3.0931	3.2727
0.3	10.0000	1.8466	1.5992	- 2.4899	0.6742	4.0489	5.0047	5.2953
0.4	8.6603	1.5992	0.9233	- 4.4016	1.1918	4.0489	5.0047	5.2953
0.5	5.0000	0.9233	0.0000	- 5.7718	1.5628	1.7554	2.1698	2.2958
0.6	0.0000	0.0000	0.0000	- 5.5955	1.5151	- 2.5024	- 3.0931	- 3.2727
0.7	0.0000	0.0000	0.0000	- 3.4582	0.9364	- 6.5513	- 8.0978	- 8.5680
0.8	0.0000	0.0000	0.0000	0.0000	0.0000	- 8.0979	- 10.0095	- 10.5906
0.9	0.0000	0.0000	0.0000	3.4582	- 0.9364	- 6.5513	- 8.0979	- 8.5680
1.0	0.0000				- 1.5151		- 3.0931	- 3.2727

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Central Difference Method

Central Difference Method (1)



□ Velocities at the time stations $t_{i-1/2}$ and $t_{i+1/2}$ are as follows

$$\dot{u}_{i-\frac{1}{2}} = \frac{u_i - u_{i-1}}{\Delta t} \quad \text{and} \quad \dot{u}_{i+\frac{1}{2}} = \frac{u_{i+1} - u_i}{\Delta t}$$

□ Then the velocity and acceleration at time station t_i are expressed as

$$\dot{u}_i = \frac{1}{2} \left(\dot{u}_{i-\frac{1}{2}} + \dot{u}_{i+\frac{1}{2}} \right) = \frac{u_{i+1} - u_{i-1}}{2\Delta t} \quad (1)$$

$$\ddot{u}_i = \frac{1}{\Delta t} \left(\dot{u}_{i+\frac{1}{2}} - \dot{u}_{i-\frac{1}{2}} \right) = \frac{u_{i+1} - 2u_i + u_{i-1}}{\Delta t^2} \quad (2)$$

Central Difference Method (2)

- Substituting Equation (1) and (2) into the equation of motion,

$$m \left(\frac{u_{i+1} - 2u_i + u_{i-1}}{\Delta t^2} \right) + c \left(\frac{u_{i+1} - u_{i-1}}{2\Delta t} \right) + k u_i = p_i$$

- Modifying the equation

$$\left[\frac{m}{\Delta t^2} + \frac{c}{2\Delta t} \right] u_{i+1} = p_i - \left[\frac{m}{\Delta t^2} - \frac{c}{2\Delta t} \right] u_{i-1} - \left[k - \frac{2m}{\Delta t^2} \right] u_i$$
$$\rightarrow k_{eff} u_{i+1} = p_{eff}$$

- Then the effective stiffness and load are defined as

$$k_{eff} = \frac{m}{\Delta t^2} + \frac{c}{2\Delta t}$$
$$p_{eff} = p_i - \left[k - \frac{2m}{\Delta t^2} \right] u_i - \left[\frac{m}{\Delta t^2} - \frac{c}{2\Delta t} \right] u_{i-1}$$

Central Difference Method (3)

$$u_{i+1} = \frac{p_{eff}}{k_{eff}}$$

Note that this is an explicit method

- The solution u_{i+1} at time $i+1$ is determined from the equilibrium condition at time i without using the equilibrium condition at time $i+1$. Such methods are called Explicit Method.

□ Step 1) Compute m , c , and k

□ Step 2) Initial conditions $u_0, \dot{u}_0, \ddot{u}_0 = \frac{p_0 - c\dot{u}_0 - ku_0}{m}$

□ Step 3) Set time step size Δt ($< T_n/\pi$, this upper limit is because of stability)

□ Step 4) Compute coefficients

$$a_0 = \frac{1}{\Delta t^2} ; a_1 = \frac{1}{2\Delta t} ; a_2 = 2a_0 ; a_3 = \frac{1}{a_2}$$

□ Step 5) Compute $u_{-1} = u_0 - \Delta t \dot{u}_0 + \frac{\Delta t^2}{2} \ddot{u}_0$

Why?

Solve for u_1 and substituting here

$$\dot{u}_0 = \frac{u_1 - u_{-1}}{2\Delta t}$$

$$\ddot{u}_0 = \frac{u_1 - 2u_0 + u_{-1}}{\Delta t^2}$$

$$u_{-1} = u_0 - \Delta t \dot{u}_0 + \frac{\Delta t^2}{2} \ddot{u}_0$$

- Step 6) Compute the effective stiffness

$$k_{eff} = a_0 m + a_1 c$$

→ Up to this step, all calculations need to be done once as initial calculations

- Step 7) For each time step, repeat the following procedures, that is,
for $i=0,1,2,\dots$

- Step 7-1) $p_{eff} = p_i - (k - a_2 m)u_i - (a_0 m - a_1 c)u_{i-1}$

- Step 7-2) $u_{i+1} = \frac{p_{eff}}{k_{eff}}$

- Step 7-3) $\dot{u}_i = a_1(u_{i+1} - u_{i-1})$
 $\ddot{u}_i = a_0(u_{i+1} - 2u_i + u_{i-1})$

- Step 7-4) Repeat for the next time step from Step 7-1)

4

Newmark Methods

□ Dynamic states of a vibrating structure can be described by

□ Displacement

□ Velocity

□ Acceleration

□ They are related through either differentiation or integration operations. For example,

$$\dot{u} = \frac{du}{dt} \quad \text{or} \quad u = \int \dot{u} dt$$

□ It is observed that with numerical differentiation, the original error presented in u will get expanded after the differential operation.

□ However, with numerical integration, the original error presented in the integrand (\dot{u}) will get smoothed out after the operation.

- This means that performing numerical integration operation is potentially of advantage in reducing or smoothing out the original error presented with the integrand. As a matter of fact, this observation serves as the basis of many engineering based temporal discretization methods. That is, instead of starting with displacements, we make assumptions on the variation of accelerations within a time step and obtain relations for velocities and displacements within the time step through numerical integration.

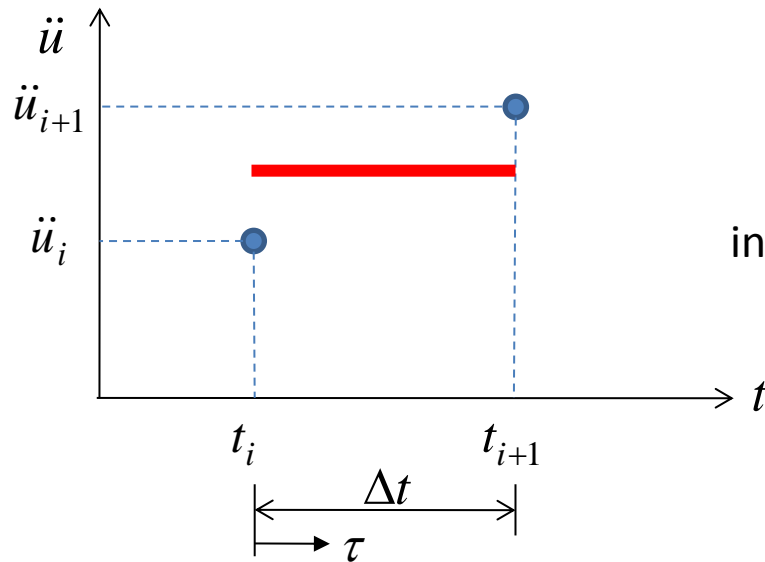
- Question) Among the three state variables (u, \dot{u}, \ddot{u}), which one is the best variable to start with for numerical integration methods?

- Answer) "Acceleration"

Newmark Method (3)

□ Let's assume variations of acceleration during Δt to be “constant” or “linear”

□ Average Acceleration (A.A.)



integrate

$$\ddot{u}(\tau) = \frac{1}{2}(\ddot{u}_{i+1} + \ddot{u}_i)$$

integrate

$$\dot{u}(\tau) = \dot{u}_i + \frac{\tau}{2}(\ddot{u}_{i+1} + \ddot{u}_i)$$

$$\tau = \Delta t$$

$$\dot{u}_{i+1} = \dot{u}_i + \frac{\Delta t}{2}(\ddot{u}_{i+1} + \ddot{u}_i)$$

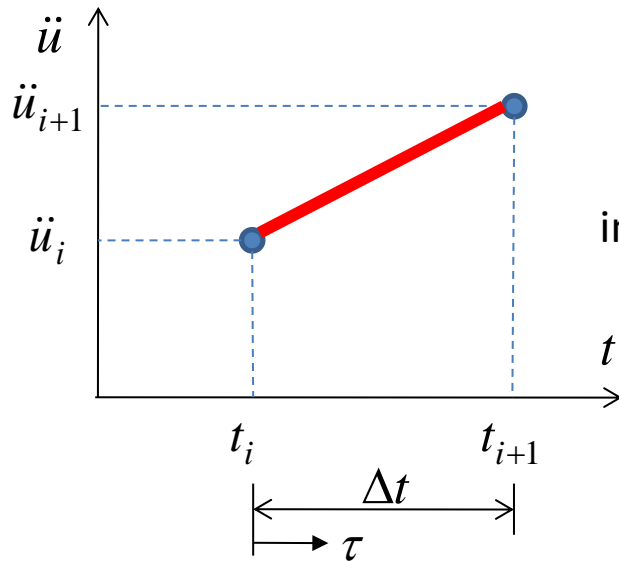
$$u(\tau) = u_i + \dot{u}_i \tau + \frac{\tau^2}{4}(\ddot{u}_{i+1} + \ddot{u}_i)$$

$$\tau = \Delta t$$

$$u_{i+1} = u_i + \dot{u}_i \Delta t + \frac{\Delta t^2}{4}(\ddot{u}_{i+1} + \ddot{u}_i)$$

Newmark Method (4)

Linear Acceleration (L.A.)



integrate

$$\ddot{u}(\tau) = \ddot{u}_i + \frac{\tau}{\Delta t} (\ddot{u}_{i+1} - \ddot{u}_i)$$

integrate

$$\dot{u}(\tau) = \dot{u}_i + \ddot{u}_i \tau + \frac{\tau^2}{2\Delta t} (\ddot{u}_{i+1} - \ddot{u}_i)$$

$$\dot{u}_{i+1} = \dot{u}_i + \ddot{u}_i \Delta t + \frac{\Delta t}{2} (\ddot{u}_{i+1} - \ddot{u}_i)$$

$$= \dot{u}_i + \frac{\Delta t}{2} (\ddot{u}_{i+1} + \ddot{u}_i)$$

$$u(\tau) = u_i + \dot{u}_i \tau + \ddot{u}_i \frac{\tau^2}{2} + \frac{\tau^3}{6\Delta t} (\ddot{u}_{i+1} - \ddot{u}_i)$$

$$u_{i+1} = u_i + \dot{u}_i \Delta t + \Delta t^2 \left(\frac{\ddot{u}_{i+1}}{6} + \frac{\ddot{u}_i}{3} \right)$$

- Introducing two integration parameters γ and β

$$\begin{aligned}\dot{u}_{i+1} &= \dot{u}_i + [(1-\gamma)\Delta t]\ddot{u}_i + (\gamma\Delta t)\ddot{u}_{i+1} \\ u_{i+1} &= u_i + (\Delta t)\dot{u}_i + \left[\left(\frac{1}{2}-\beta\right)\Delta t^2\right]\ddot{u}_i + [\beta(\Delta t^2)]\ddot{u}_{i+1}\end{aligned}\tag{3}$$

- A.A. $\rightarrow \gamma = \frac{1}{2} ; \beta = \frac{1}{4}$

- L.A. $\rightarrow \gamma = \frac{1}{2} ; \beta = \frac{1}{6}$

Formulation of Newmark Method for Linear System (1)

- Using incremental formulation,

$$\Delta u_i \equiv u_{i+1} - u_i \quad ; \quad \Delta \dot{u}_i \equiv \dot{u}_{i+1} - \dot{u}_i$$

$$\Delta \ddot{u}_i \equiv \ddot{u}_{i+1} - \ddot{u}_i \quad ; \quad \Delta p_i \equiv p_{i+1} - p_i$$

$$m\Delta \ddot{u}_i + c\Delta \dot{u}_i + k\Delta u_i = \Delta p_i \quad \text{-----} \quad (4)$$

- Rewriting Equation (3)

$$\Delta \dot{u}_i = \dot{u}_i \Delta t + (\gamma \Delta t) \Delta \ddot{u}_i$$

Solve for $\Delta \ddot{u}_i$ and substituting into first equation

$$\Delta u_i = \dot{u}_i \Delta t + \frac{\Delta t^2}{2} \ddot{u}_i + \beta \Delta t^2 \Delta \ddot{u}_i$$

$$\Delta \ddot{u}_i = \frac{1}{\beta \Delta t^2} \Delta u_i - \frac{1}{\beta \Delta t} \dot{u}_i - \frac{1}{2\beta} \ddot{u}_i$$

$$\Delta \dot{u}_i = \frac{\gamma}{\beta \Delta t} \Delta u_i - \frac{\gamma}{\beta} \dot{u}_i + \Delta t \left(1 - \frac{\gamma}{2\beta} \right) \ddot{u}_i$$

$$\left. \begin{array}{l} \Delta \ddot{u}_i = \frac{1}{\beta \Delta t^2} \Delta u_i - \frac{1}{\beta \Delta t} \dot{u}_i - \frac{1}{2\beta} \ddot{u}_i \\ \Delta \dot{u}_i = \frac{\gamma}{\beta \Delta t} \Delta u_i - \frac{\gamma}{\beta} \dot{u}_i + \Delta t \left(1 - \frac{\gamma}{2\beta} \right) \ddot{u}_i \end{array} \right\} \text{-----} (5)$$

↓
Substituting these to Equation (4)

Formulation of Newmark Method for Linear System (2)

- Then, the equilibrium equation becomes

$$k_{eff} \Delta u = \Delta p_{eff}$$

$$\Delta u = \frac{\Delta p_{eff}}{k_{eff}}$$

Where

$$k_{eff} = k + \frac{\gamma}{\beta \Delta t} c + \frac{1}{\beta \Delta t^2} m$$

$$\Delta p_{eff} = \Delta p_i + \left(\frac{1}{\beta \Delta t} m + \frac{\gamma}{\beta} c \right) \dot{u}_i + \left[\frac{1}{2\beta} m + \Delta t \left(\frac{\gamma}{2\beta} - 1 \right) c \right] \ddot{u}_i$$

- Once we know $\Delta u_i = u_{i+1} - u_i$, \ddot{u}_{i+1} and \dot{u}_{i+1} can be computed from Equation (5)

Formulation of Newmark Method for Linear System (3)

- However, it is recommended to compute \ddot{u}_{i+1} from the equilibrium equation at time t_{i+1} .

$$\ddot{u}_{i+1} = \frac{p_{i+1} - c\dot{u}_{i+1} - ku_{i+1}}{m}$$

- Remarks: For nonlinear problems, use this equation for \ddot{u}_{i+1} to ensure dynamic equilibrium at the end of the time step

Computational Procedures for Newmark Method (1)

□ Choose one

$$\left\{ \begin{array}{l} \text{A.A. } \gamma = \frac{1}{2} ; \beta = \frac{1}{4} \\ \text{L.A. } \gamma = \frac{1}{2} ; \beta = \frac{1}{6} \end{array} \right.$$

□ Step 1) Compute m , c , and k

□ Step 2) Initial conditions $u_0, \dot{u}_0, \ddot{u}_0 = \frac{p_0 - c\dot{u}_0 - ku_0}{m}$

□ Step 3) Set time step size Δt

□ Step 4) Compute coefficients

$$a_0 = \frac{\gamma}{\beta \Delta t} ; a_1 = \frac{1}{\beta \Delta t^2} ; a_2 = \frac{1}{\beta \Delta t}$$

$$a_3 = \frac{\gamma}{\beta} ; a_4 = \frac{1}{2\beta} ; a_5 = \Delta t \left(\frac{\gamma}{2\beta} - 1 \right)$$

Computational Procedures for Newmark Method (2)

- Step 5) Compute the effective stiffness

$$k_{eff} = k + a_0 c + a_1 m$$

→ Up to this step, all calculations need to be done once as initial calculations

- Step 6) For each time step, repeat the following procedures, that is, for $i=0,1,2,\dots$

- Step 6-1) $\Delta p_{eff} = \Delta p_i + (a_2 m + a_3 c) \dot{u}_i + (a_4 m + a_5 c) \ddot{u}_i$

- Step 6-2) $\Delta u_i = \frac{\Delta p_{eff}}{k_{eff}}$

- Step 6-3) $\Delta \dot{u}_i = a_0 \Delta u_i - a_3 \dot{u}_i - a_5 \ddot{u}_i$

- Step 6-4) $u_{i+1} = u_i + \Delta u_i \quad ; \quad \dot{u}_{i+1} = \dot{u}_i + \Delta \dot{u}_i$

- Step 6-5) $\ddot{u}_{i+1} = \frac{p_{i+1} - c \dot{u}_{i+1} - k u_{i+1}}{m}$

Computational Procedures for Newmark Method (3)

□ **Step 7)** Repeat for next time step

□ Remarks on Stability Conditions

□ For A.A. $\gamma = \frac{1}{2}$; $\beta = \frac{1}{4}$

$$\boxed{\frac{\Delta t}{T_n} < \infty} \longrightarrow \text{A.A. is unconditionally stable}$$

□ For L.A. $\gamma = \frac{1}{2}$; $\beta = \frac{1}{6}$

$$\boxed{\frac{\Delta t}{T_n} < 0.551} \longrightarrow \text{L.A. is conditionally stable}$$

□ Note that in case of multi DOF systems, stability matters since T_n in the criteria is the natural period of the highest mode

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Stability and Computational Errors

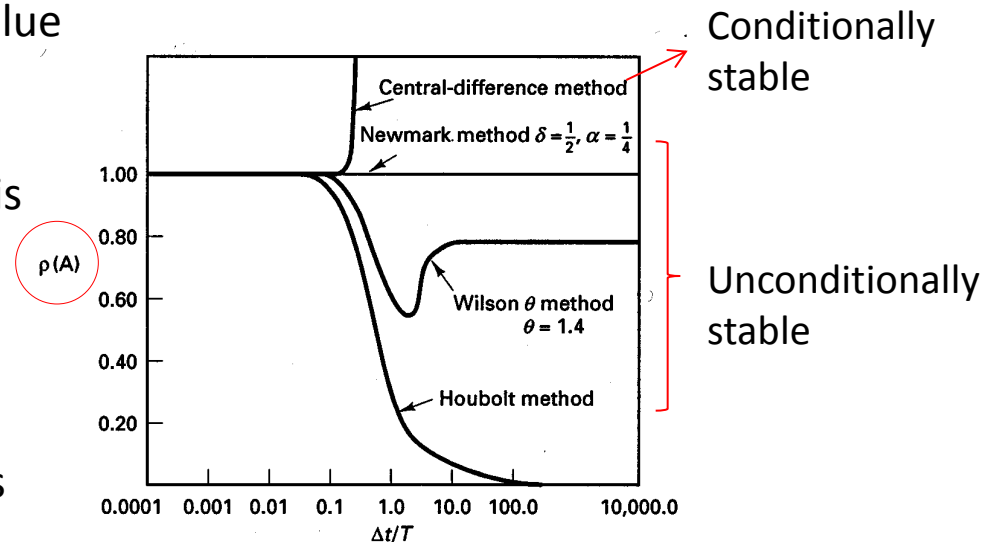
□ Stability

- Numerical procedures that lead to bounded solutions if the time step is shorter than some stability limit are called conditionally stable procedures.
- Procedures that lead to bounded solutions regardless of the time-step length are called unconditionally stable procedure.
- The average acceleration method is unconditionally stable.
- The linear acceleration method is stable if $\Delta t / T_n < 0.551$.
- The central difference method is stable if $\Delta t / T_n < 1/\pi$.

Spectral radius($\rho(A)$) is a modulus of eigenvalue of operator matrix, A .

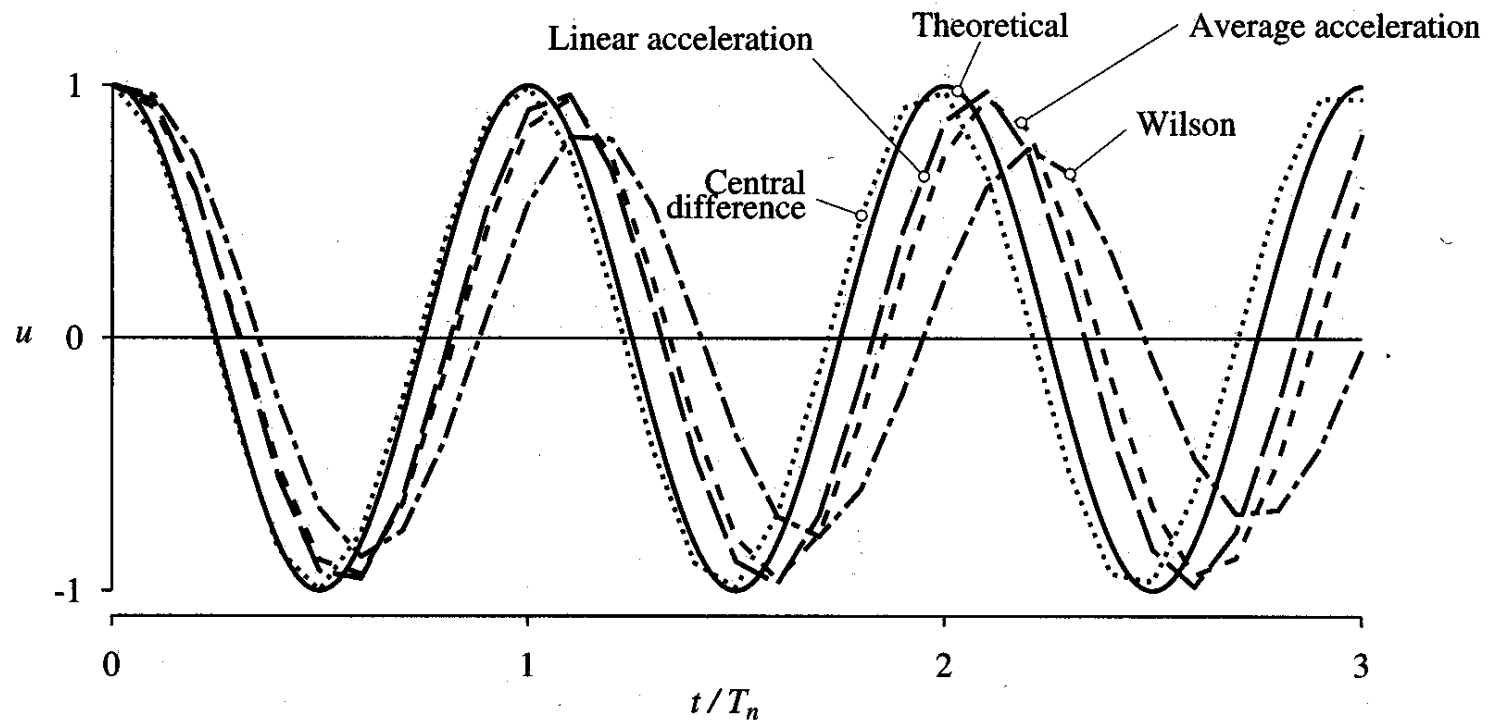
If $\rho(A)$ this is less than 1, then the method is unconditionally stable like Newmark method (A.A.), Wilson- θ method and Houbolt method.

If $\rho(A)$ this is greater than 1, the method is conditionally stable like CDM.



Stability and Computational Errors (2)

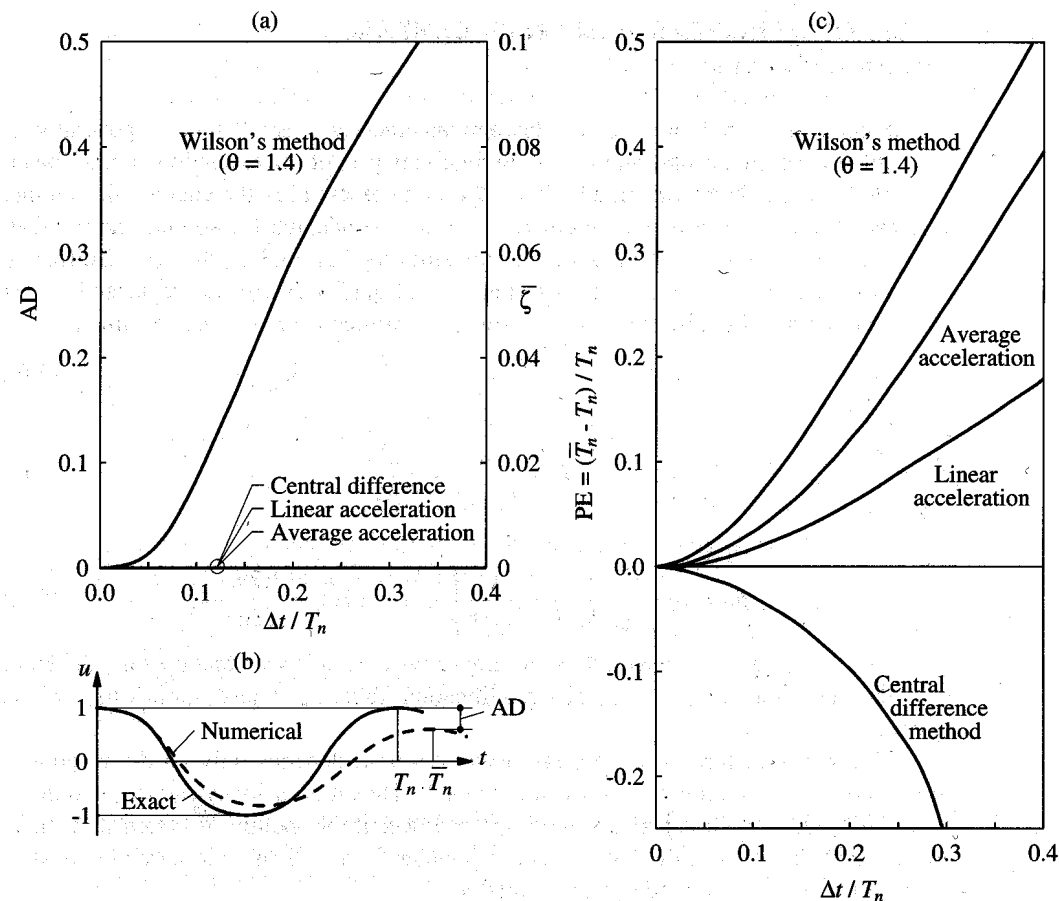
- The figure below shows that some numerical methods may predict that the displacement amplitude decays with time, although the system is undamped, and that the natural period is elongated or shortened.



□ The figure below shows the AD (amplitude decay) and PE (period elongation) in the four numerical methods as a function of $\Delta t / T_n$.

□ The AD in Wilson's method implies that this method introduces numerical damping.

□ The central difference method introduces the largest period error and has stability limit of $\Delta t / T_n = 1/\pi$.



Stability and Computational Errors (4)

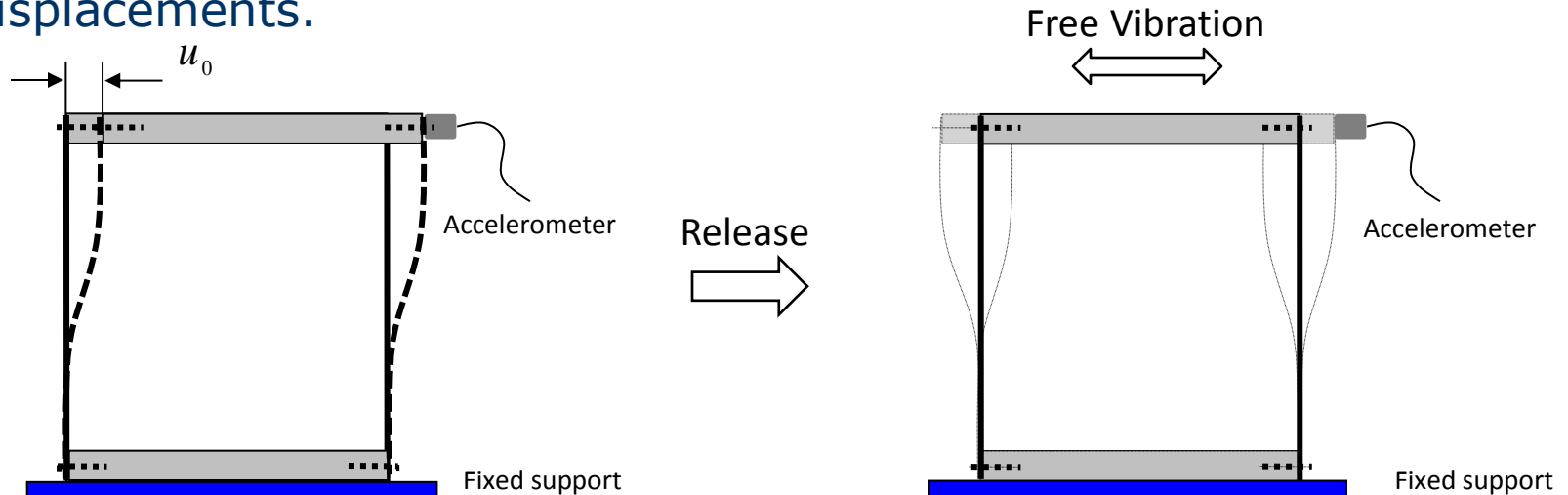
- ❑ The choice of time step also depends on the time variation of the dynamic excitation, in addition to the natural vibration period of the system.
 - ❑ Figure (c) on the previous page suggests that $\Delta t = 0.1 T$ would give reasonably accurate results.
 - ❑ The time step should also be short enough to keep the distortion of the excitation function to a minimum.

6

Hands-on Experiment Project

Problem Statement (1)

- ❑ **Conduct free vibration test** of an one-story shear building structure and **measure accelerations under free vibration**, and integrate accelerations to get experimental velocities and displacements.



- ❑ **Implement the time integration methods** explicitly presented in this lecture note. MATLAB programming language is recommended.
- ❑ Using the time integration methods (Central difference method and Newmark methods), **compute numerical responses (accelerations, velocities and displacements) and compare with the measured dynamic responses.**

Problem Statement (2)

Note) The NI LabVIEW VI program automatically determines the initial displacement and velocity to be used in subsequent numerical computations by time integration methods. (*Please note that the initial displacement does not need to be actual u_0 which is applied in the lab test. The initial displacement could be the one at any moment when the actual measuring starts.*)

Important Notes for Test

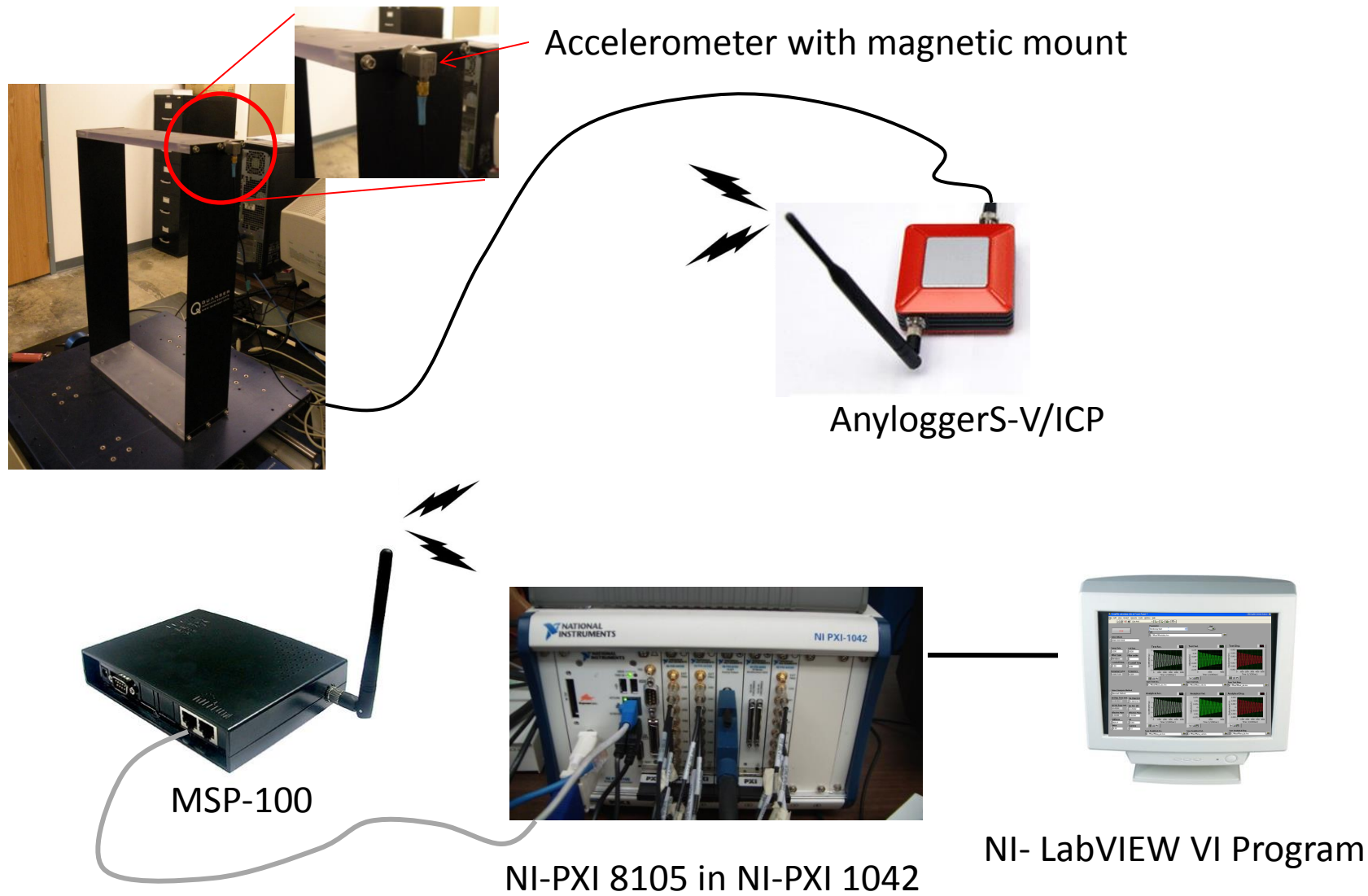
- ❑ Please follow procedures in “**8. Users manual for NI LabVIEW VI programs**”.
- ❑ You are provided with the geometrical dimension of the test structure as well as the calibration information for the sensors. Therefore, you can compute the effective stiffness. The mass of floor of the structure (including the accelerometer and mounting plate) is 1550.0 gram. Also, sensor readings are given in gravities (**$1g=9.8 \text{ m/sec}^2$**). The saturation level (maximum reading) for the two accelerometers is +/- 50g (it depends on the sensors to be used). Note that in some cases there may be an offset in the accelerometer. However, the saved acceleration data has no DC-offset because the NI-LabVIEW program automatically removes the DC-offset before saving.
- ❑ The NI-LabVIEW VI program will automatically determine the effective mass value, the natural frequency, and the damping factor based on the logarithmic decrement method and the provided stiffness value.

- ❑ Write a full report using the instructions provided in class. Organize your report into sections (e.g. Introduction, Procedures, Results, Discussion, Summary, References). Write concisely and clearly.
 - ❑ Include the following: (1) A schematic diagram and description of the test equipment. (2) Hand-calculated effective stiffness. (3) Natural frequency, and damping factor measured from test. (4) Free-body diagrams, and the equation of motion for your mathematical model. (5) Plots of the free vibration responses measured from the experiment. (6) Plots of the numerical responses computed by the NI-LabVIEW VI program. (7) Plots of the numerical responses computed by your own codes. and (8) Assumptions and explanations for differences between numerical responses and test measurements.
 - ❑ Make plots of free vibration responses from experiments and numerical computations on **the same graphs** so that they are explicitly compared.
-

7

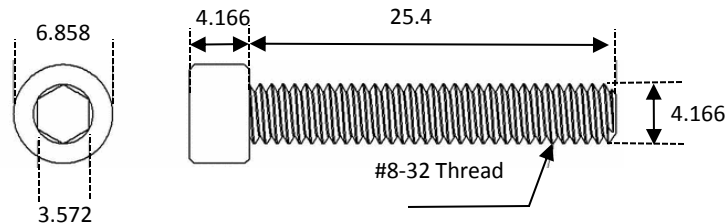
Test Setup and Equipment

Test Setup

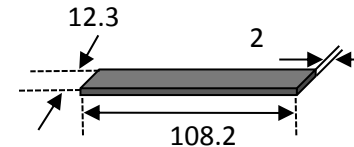


Test Structure

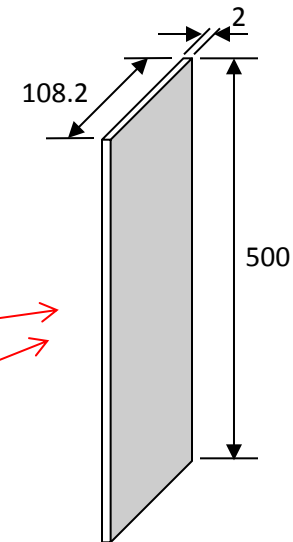
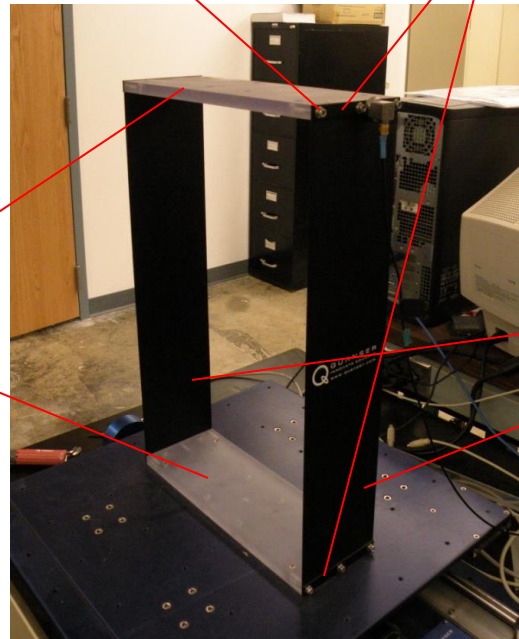
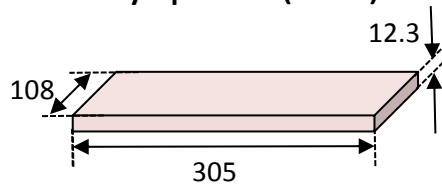
Bolts (12EA)



Steel plate (4EA)



Acryl plate (2EA)

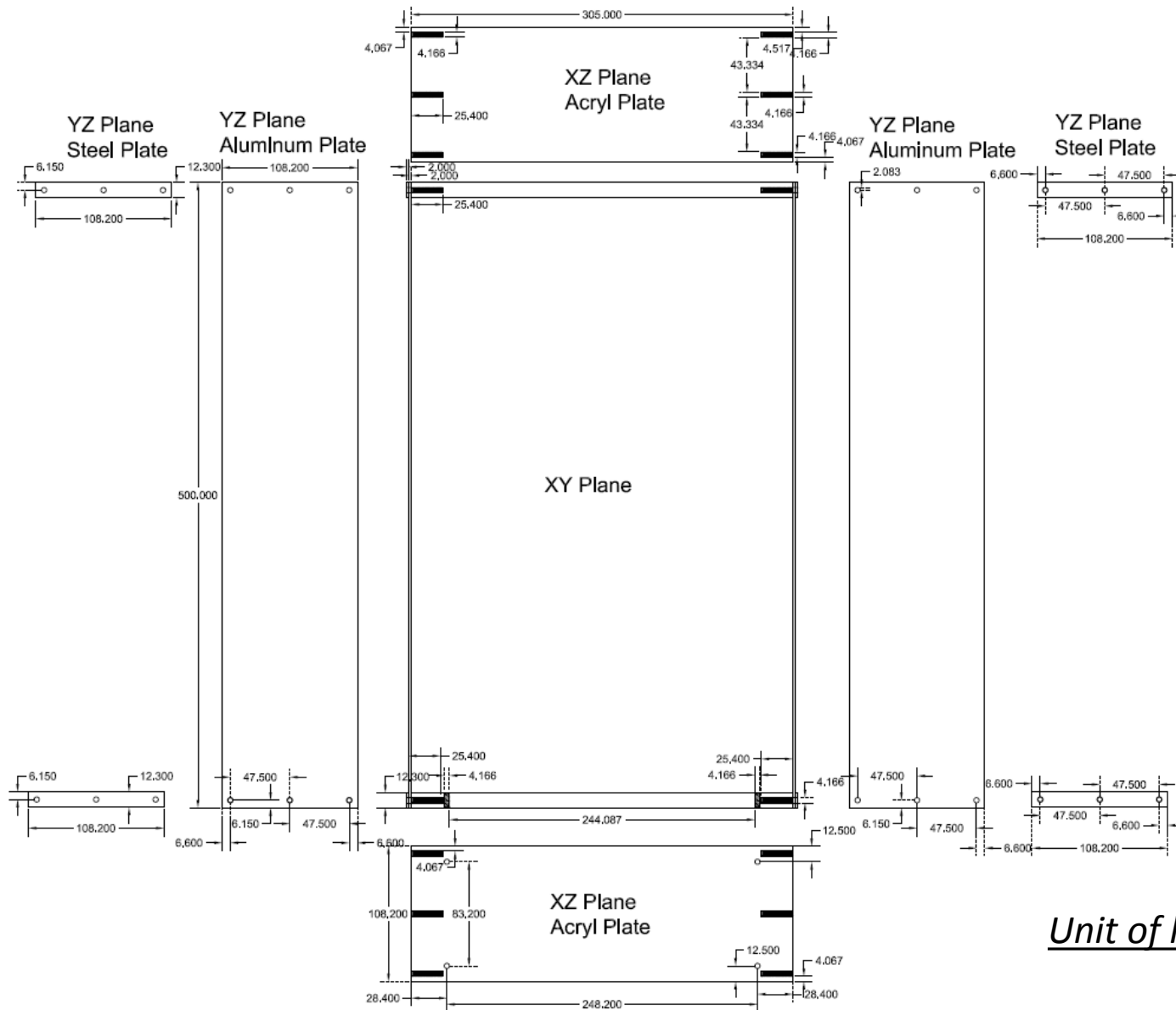


Aluminum plate (2EA)
($E=70$ Gpa)

Total weight :1550.0 g (including sensor)

Unit of length : mm

Test Structure- Drawing



Unit of length : mm

Test Equipment

- ❑ NI-PXI 8105 Controller
- ❑ ICP type Dytran triaxial accelerometer (3093B1)
- ❑ One-channel AnyLogger (Korea Maintenance Co., LTD): AnyLoggerS-V/ICP.
- ❑ NI-LabVIEW 8.6

Specification

- ❑ Intel Core Duo Processor T2500(2.0 GHz dual core)
- ❑ 512 MB (1 x 512 MB DIMM) dual channel 667 MHz DDR2 RAM standard, 4 GB (2 x 2 GB DIMMs) maximum
- ❑ Integrated I/O
 - ❑ 10/100/1000BASE-TX Ethernet
 - ❑ 4 Hi-Speed USB ports
 - ❑ ExpressCard/34 slot
 - ❑ DVI-I video connector
 - ❑ GPIB (IEEE 488) controller
 - ❑ RS232 serial port
 - ❑ IEEE 1284 ECP/EPP parallel port
 - ❑ Integrated hard drive
- ❑ Internal PXI trigger bus routing
- ❑ Watchdog timer Software
- ❑ Hard drive-based recovery image PXI System



Dytran Accelerometer (3093B1)



Model 3093B1 Dytran Triaxial Accelerometer

Specification	Value	Unit
Weight	13.5	Grams
Size(Height x Width x Depth)	0.54 x 0.59 x 0.59	Inch
Sensitivity	100	mV/G
Ranges	+/-50	G
Frequency Response	2 to 5000	Hz
Equivalent Electrical Noise	0.007	G, RMS
Linearity	1	% F.S.
Temp. Range	-60 ~ +250	°F
Supply Current Range each axis	2 to 20	mA
Supply Voltage Range each axis	+18 to +30	VDC
Output impedance	100	OHMS



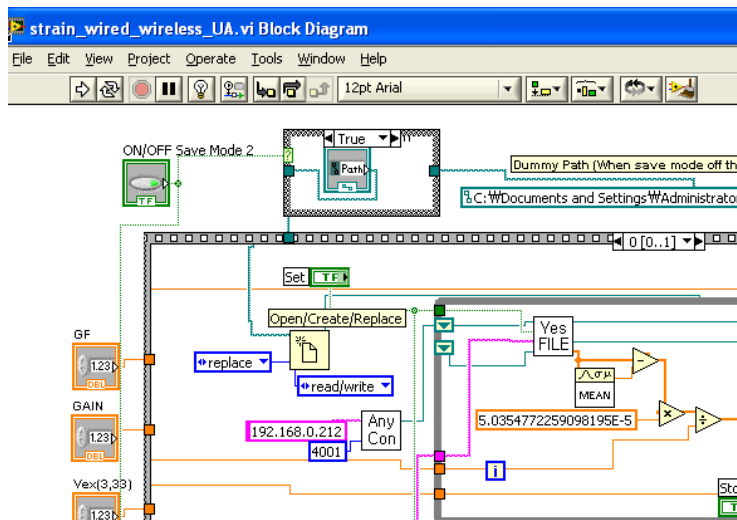
AnyloggerS-V/ICP for acceleration transmitter
AnyLoggerS-B for strain transmitter

Specification

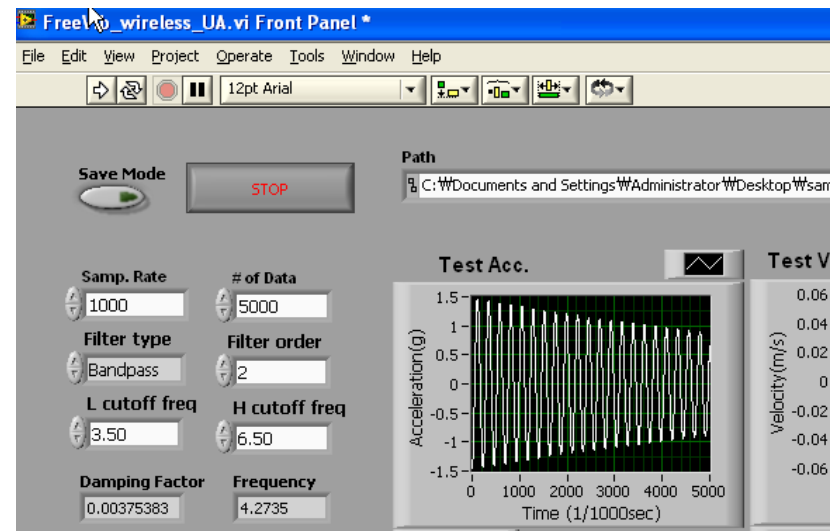
Contents	AnyloggerS-V/ICP	AnyLoggerS-B
Support Num. of Channel	1	1
Input Voltage Range	-5 ~ 5V	0 ~ 3V
Gain	1,2,5,10,20,50,100	50,100,250,500,1000,2500,5000
Programmable	10~1000Hz (10,20,50,100,200,500,1000)	10 ~ 1000Hz
Lower Pass Filter		10,20,50,100,200,500,1000
Prog. Offset	0 ~ 5V(12Bit)	0 ~ 3.3V(12Bit)
Max Sampling Rate	1000Hz	1000Hz
Exciting Voltage	24V(Only ICP Type)	3.3V±0.5%
Connector	BNC Connector	4Pin Circular Connector
SIM Usage	No	Yes
Power Consumption (w/o sensor)	150mA	100mA(w/o SIM)
Internal Battery	Li-ion Rechargeable 1500mAh x 2EA(Serial)	Li-ion Rechargeable 1500mAh x 2EA(Serial)
Ext. Power Requirement	5V	5V
Operation Temperature	-10 ~ 80℃	-10 ~ 80℃
Sync. Accuracy	< 10ms	< 10ms
Dimension	800 x 973 x 353	800 x 883 x 353
Weight	210g	210g
ADC Resolution	Differential 16Bit	Differential 16Bit
Measurement Accuracy	F.S. 0.1%	F.S. 0.1%
Sensor Connectivity	Voltage or ICP source	Bridge type sensor
Signal Ripple	Depends on Gain	Depends on Gain
Communication	Bluetooth v1.2 class1 18dBm(w/o ant.)	Bluetooth v1.2 class1 18dBm(w/o ant.)
Radio Frequency Range	2.402 ~ 2.480GHz	2.402 ~ 2.480GHz
Transmission Method	FHSS(freq. Hopping Spread Spectrum)	FHSS
Modulation Method	GFSK(Gaussian-filtered Freq. Shift Keying)	GFSK
Approvals	MIC, FCC, CE	MIC, FCC, CE

❑ LabVIEW (Laboratory Virtual Instrument Engineering Workbench) is a graphical programming language that uses icons instead of lines of text to create applications.

❑ In contrast to text-based programming languages, where instructions determine the order of program execution, LabVIEW uses dataflow programming, where the flow of data through the nodes on the block diagram determines the execution order of the VIs and functions. VIs, or virtual instrument, are LabVIEW programs that imitate physical instruments.



VI Block Diagram



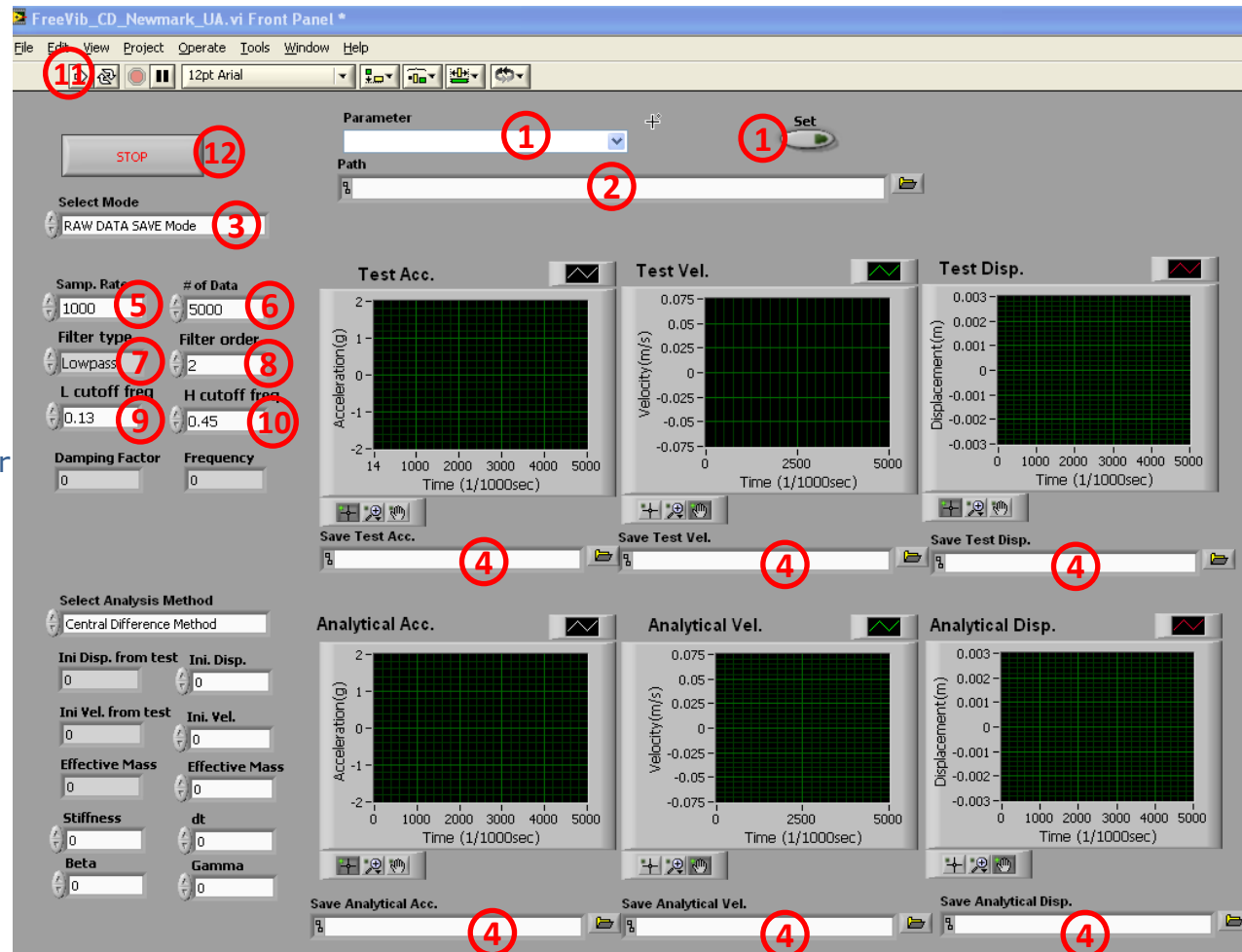
Front Panel

8

Users Manual of NI-LabVIEW VI Program

Users Manual of LabVIEW VI Prog. (1)

- 1) **Set Parameter** – to “Monitoring Start” and click “Set”
- 2) **Set Path** – Create empty files in “Path” where the original raw Acc. data will be saved
- 3) **Select Mode “RAW DATA SAVE MODE”** – Save the original raw data to the file
- 4) **Set files-** Create files where measured and computed data will be saved and link them to this panel by choosing them.
- 5) **Sampling Rate** – Default 1KHz
- 6) **Number of data** – Input the number of data for display and analysis
- 7) **Set Filter Type** – Choose “Bandpass”
- 8) **Set Filter Order**
- 9) **Set Low Cutoff Freq**
- 10) **Set High Cutoff Freq**
- 11) **Run-** Run the front panel while the structure is vibrating
- 12) **Stop** – After more than specified times (# of Data/ Samp. Rate), click “Stop”. Users can control the monitoring time.



Users Manual of LabVIEW VI Prog. (2)

During testing, measured raw data will be saved. Next steps are for computing Test Acc., Test Vel. and Test Disp. based on saved raw data.

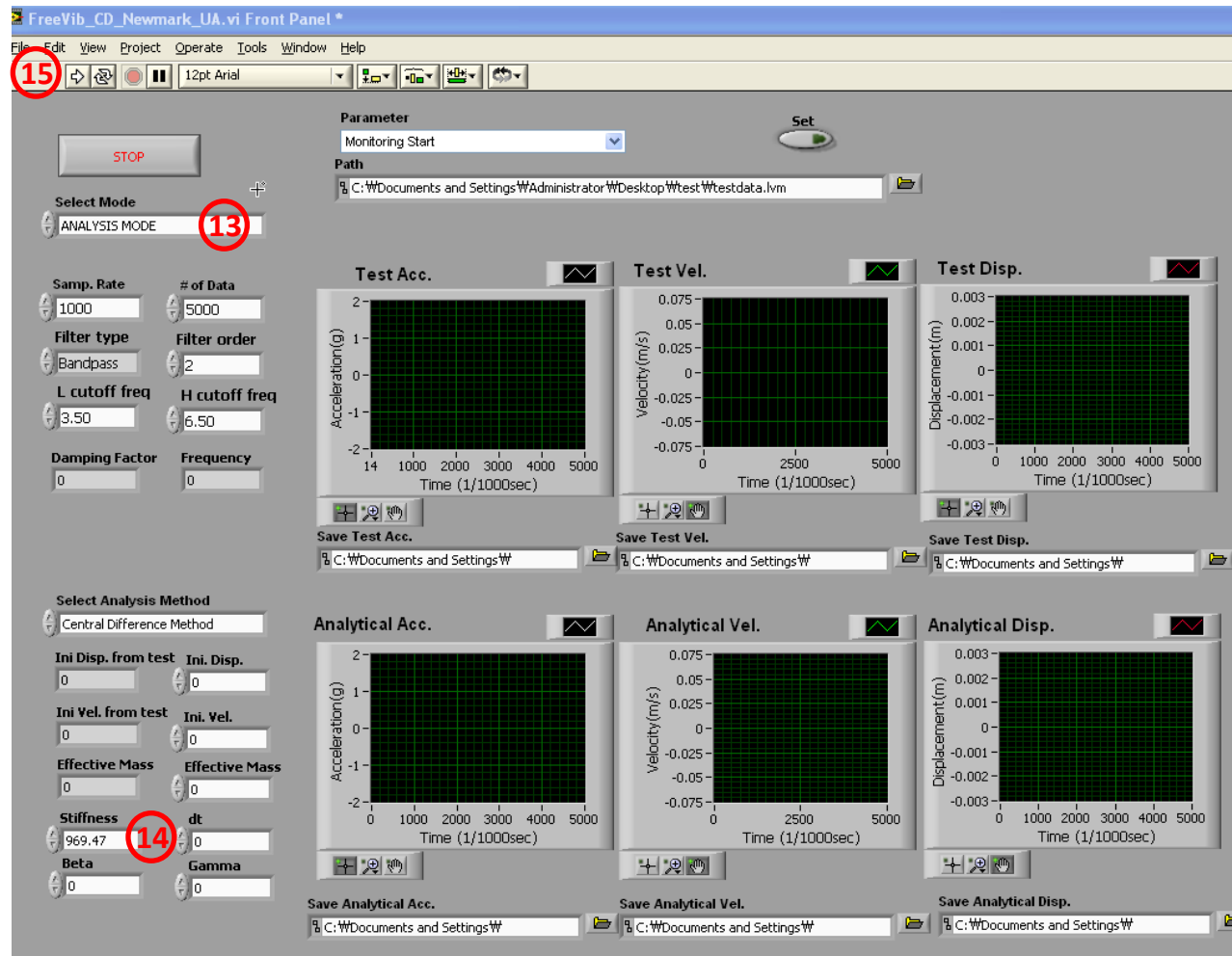
And in this step, Ini. Disp., Ini. Vel. and effective mass will be presented automatically from the test data and hand calculated stiffness value.

13) Select Mode "ANALYSIS MODE"
– Compute test Acc., Vel. and Disp. and save all the computed data to the file

14) Stiffness– Input stiffness of the test structure to get the effective mass of the vibration system using the following equation.

$$m_{eff} = \frac{k}{(2\pi f_n)^2}$$

15) Run– Run the front panel. Test Acc., Test Vel. and Test Disp. will be plotted. And Ini. Disp., Ini. Vel., and Effective Mass will be displayed as shown in the next slide.



Users Manual of LabVIEW VI Prog. (3)

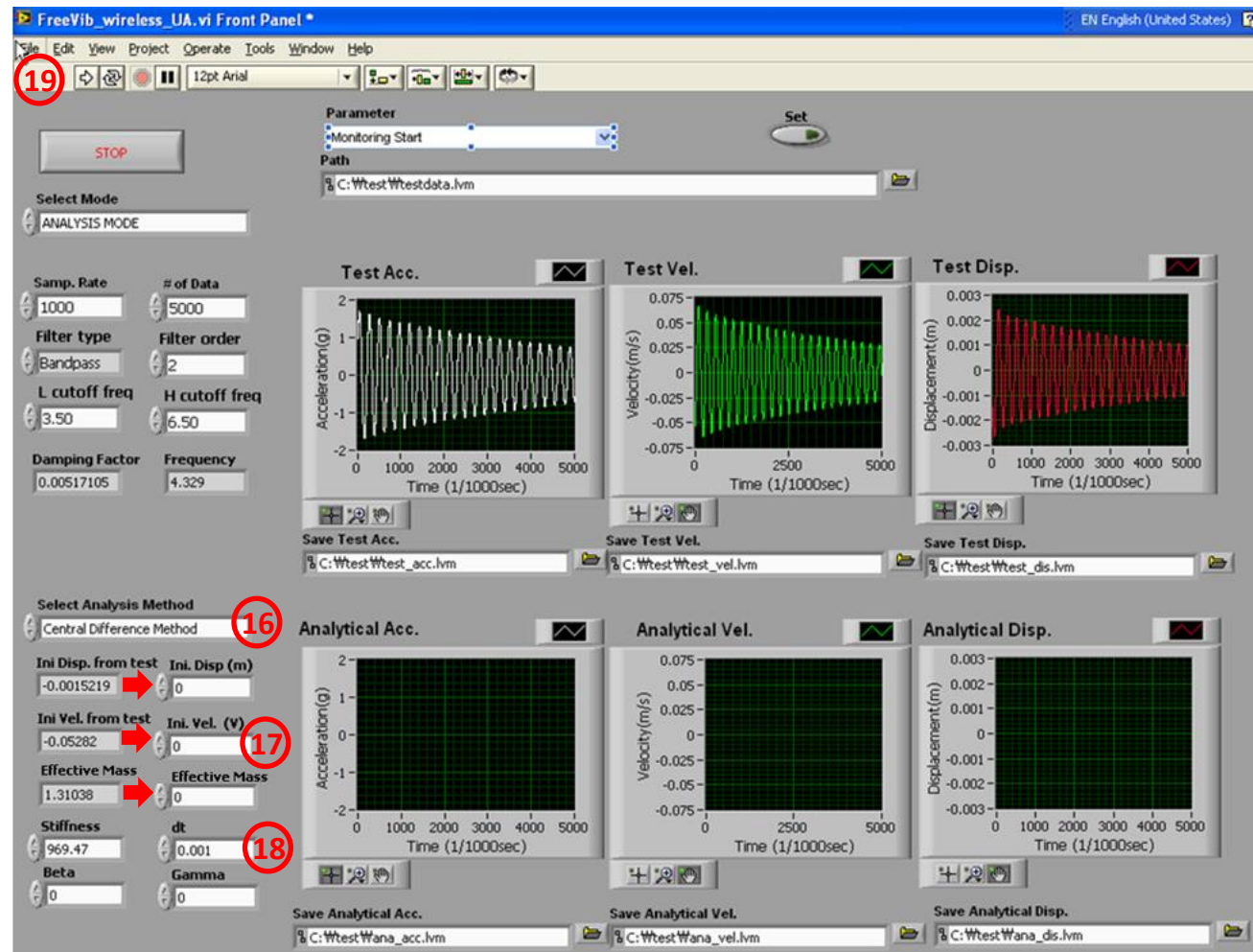
Next step is to run Central Difference Method to simulate the analytical Acc., Vel. and Disp.

16) Select Analysis Method "Central Difference Method"

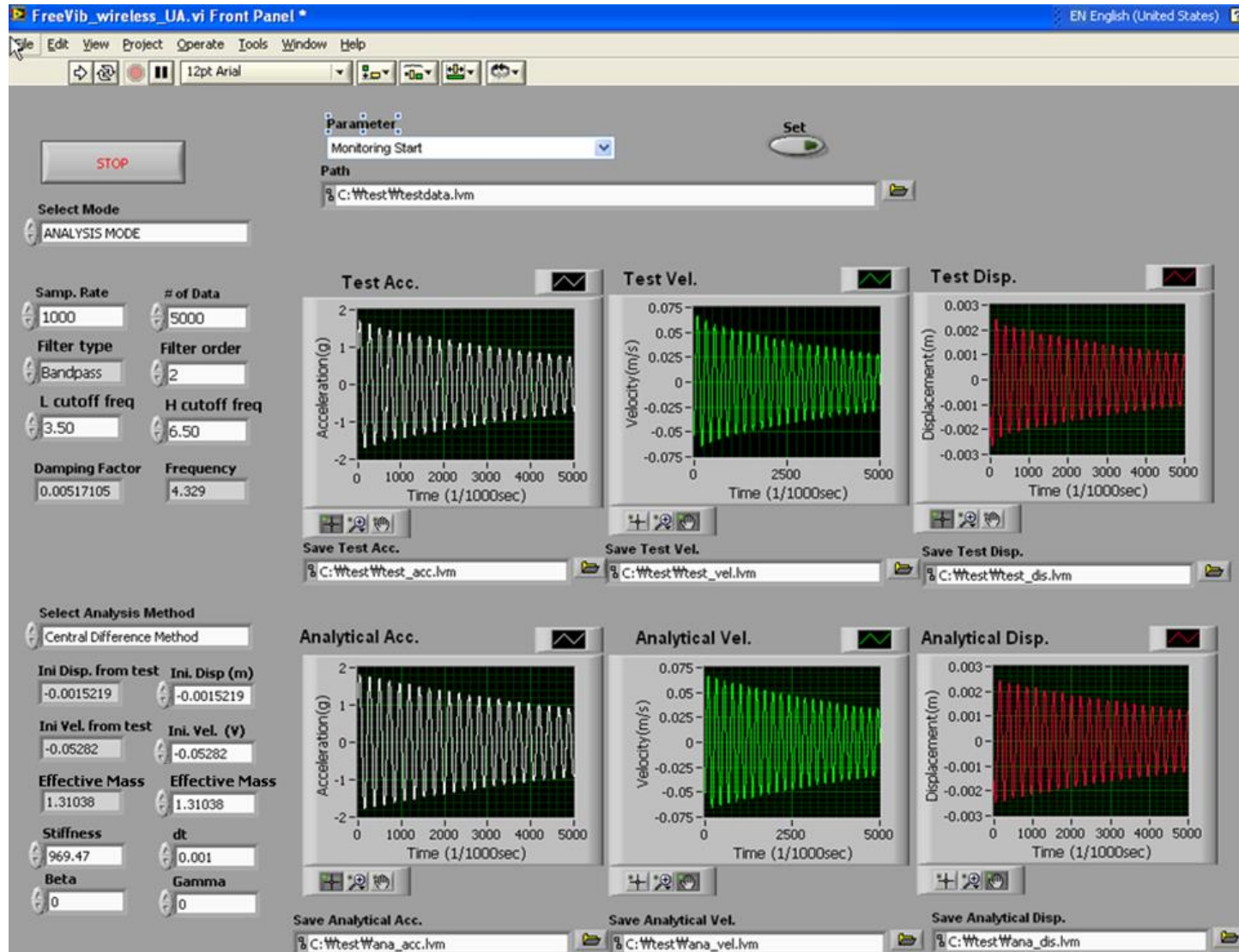
17) **Copy Data** – Copy and paste measured Ini. Disp. Ini. Vel. and automatically computed effective mass to corresponding empty text box.

18) **Delta t** – Input dt

19) **Run**- Run the front panel. Wait until computed responses are plotted. It may takes several minutes depending on the system.



Results of Central Difference Method



Users Manual of LabVIEW VI Prog. (5)

These steps are to run Newmark Method to simulate the numerical Acc., Vel. and Disp.

16) Select Analysis Method "Newmark Method"

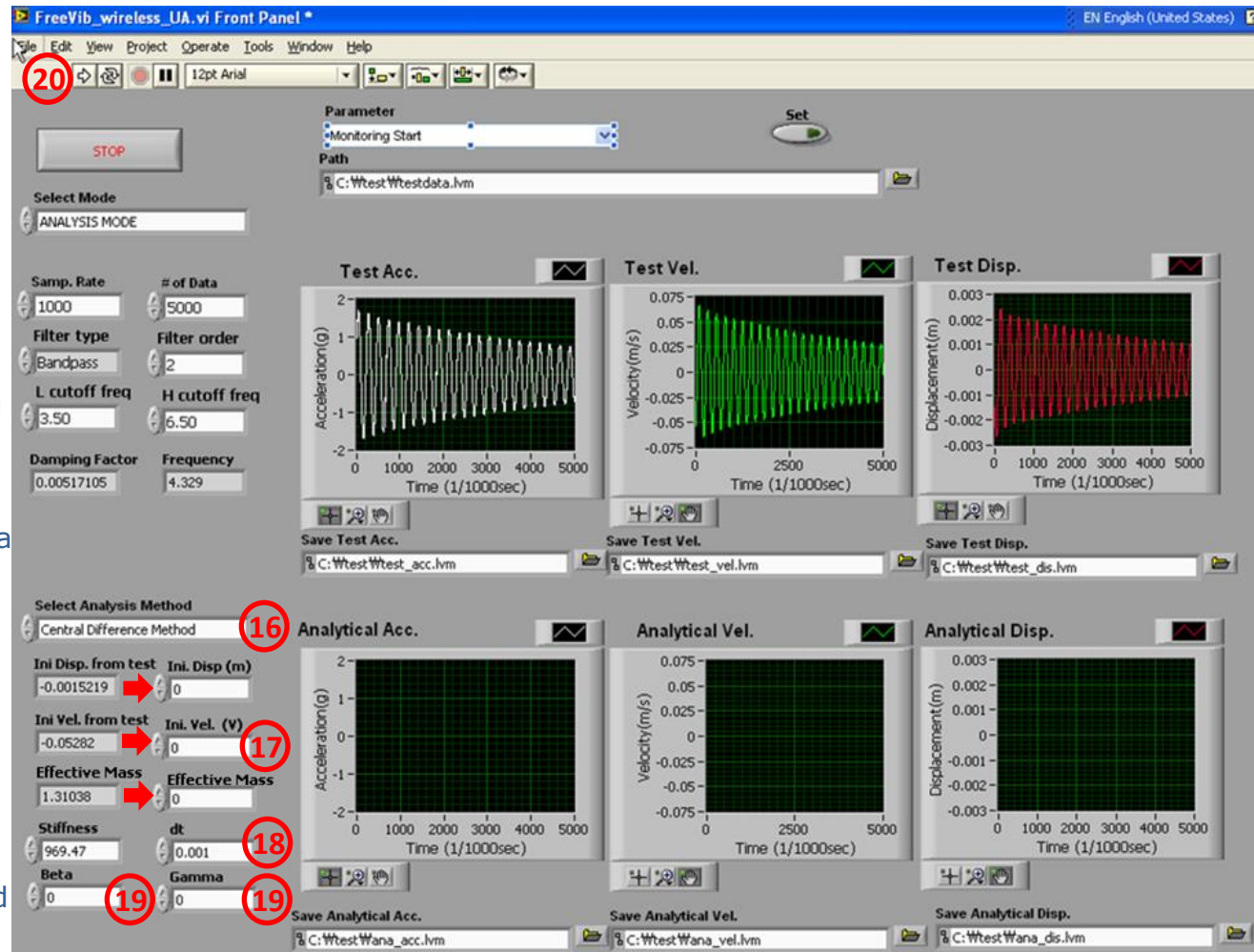
17) Copy Data – Copy and paste measured Ini. Disp., Ini. Vel. and automatically computed effective mass to corresponding empty text box.

18) Delta t – Input dt

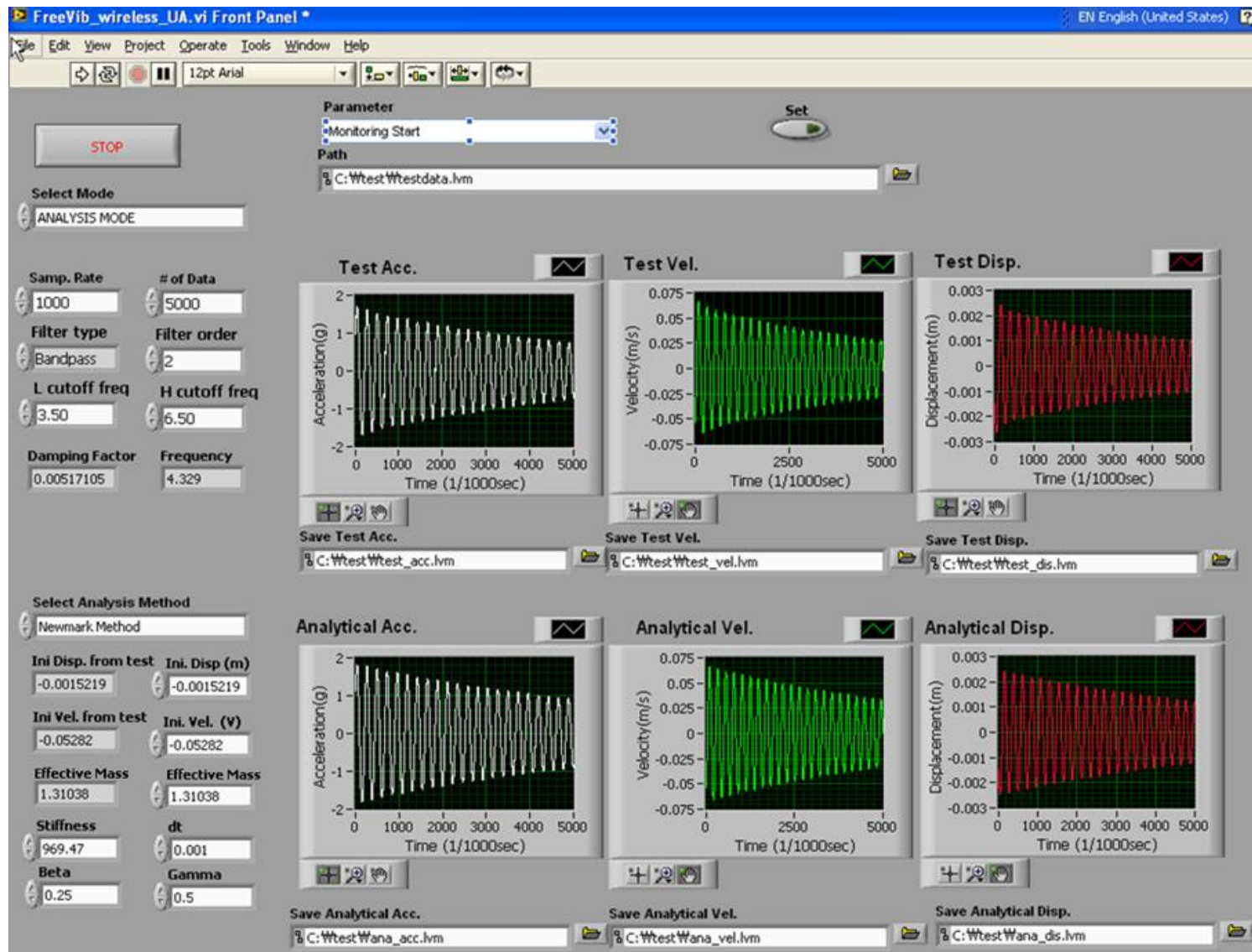
19) Beta & Gamma - Input beta and gamma values (default beta and gamma is 0.25 and 0.5, respectively for A.A.)

20) Run- Run the front panel. Wait until computed responses are plotted. It may take several minutes depending on the system.

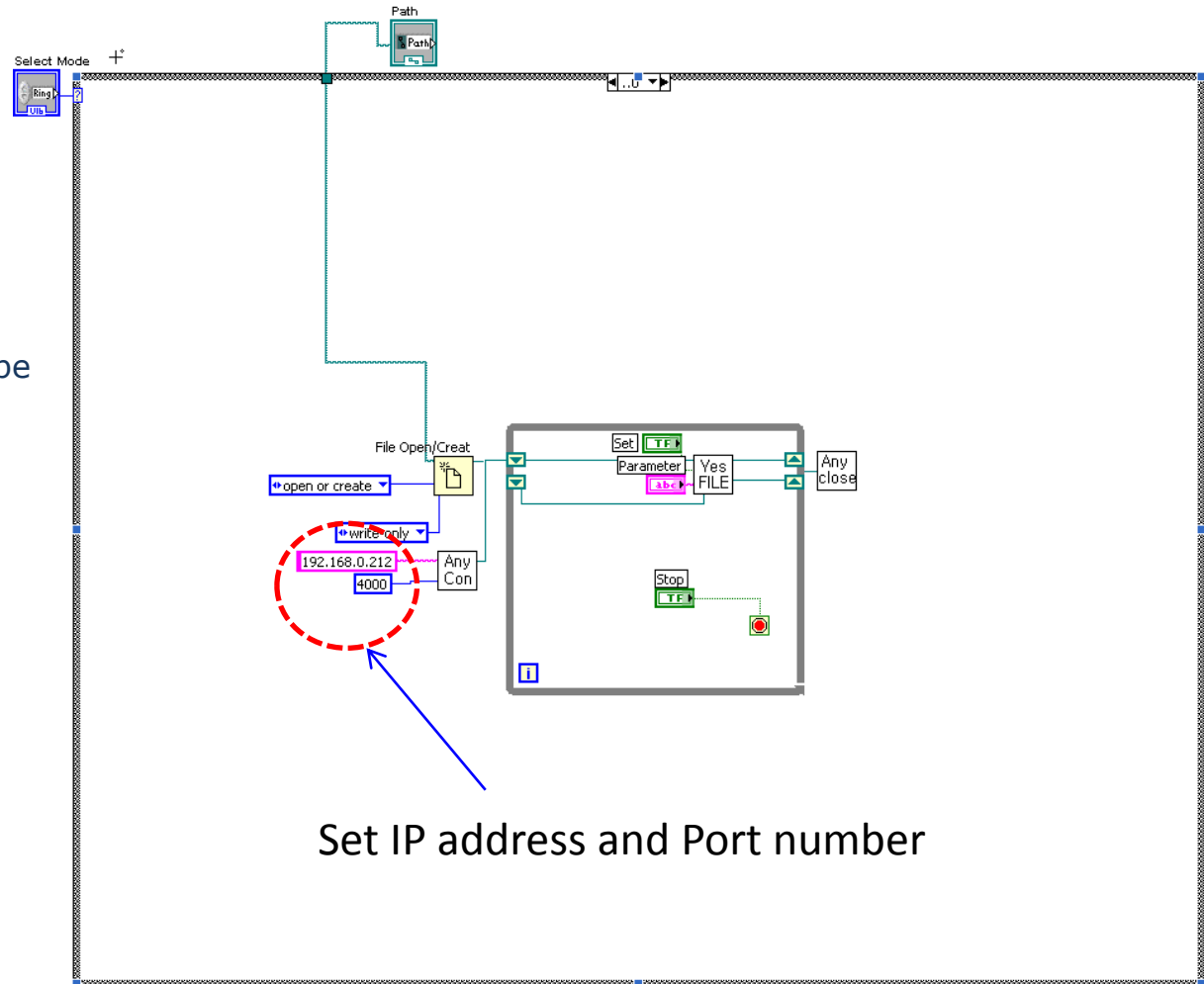
Note that Newmark method can be run right after CD Method analysis and CD method also can be run right after Newmark method analysis.



Results of Newmark Method (Average Acceleration Method)

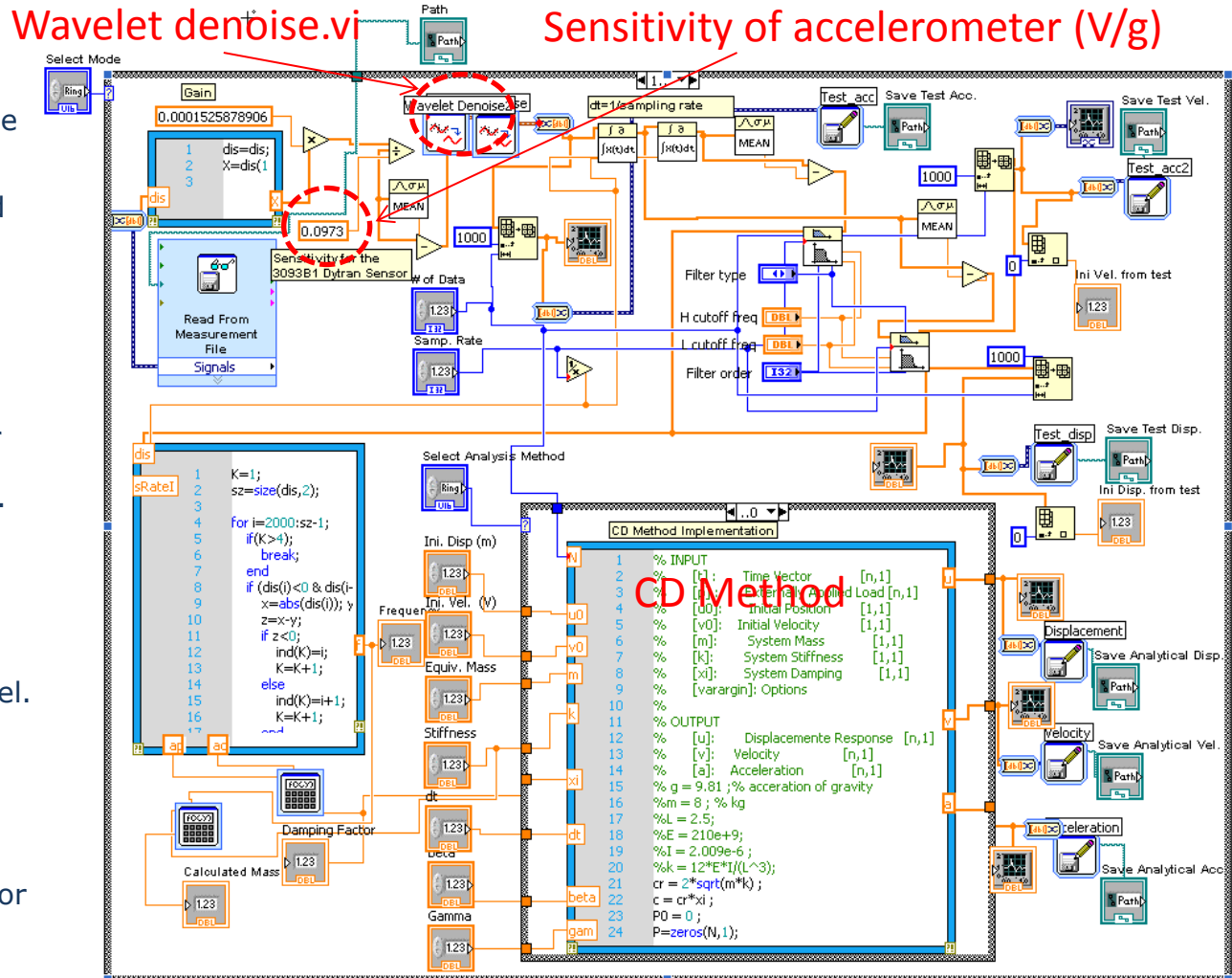


- ❑ Two-stack sequences were used
- ❑ In the first frame, users should define correct **IP address** and **port number** based on the equipments following the ANYLOGGER ® manual.
- ❑ In this frame, the raw data will be saved at the given path



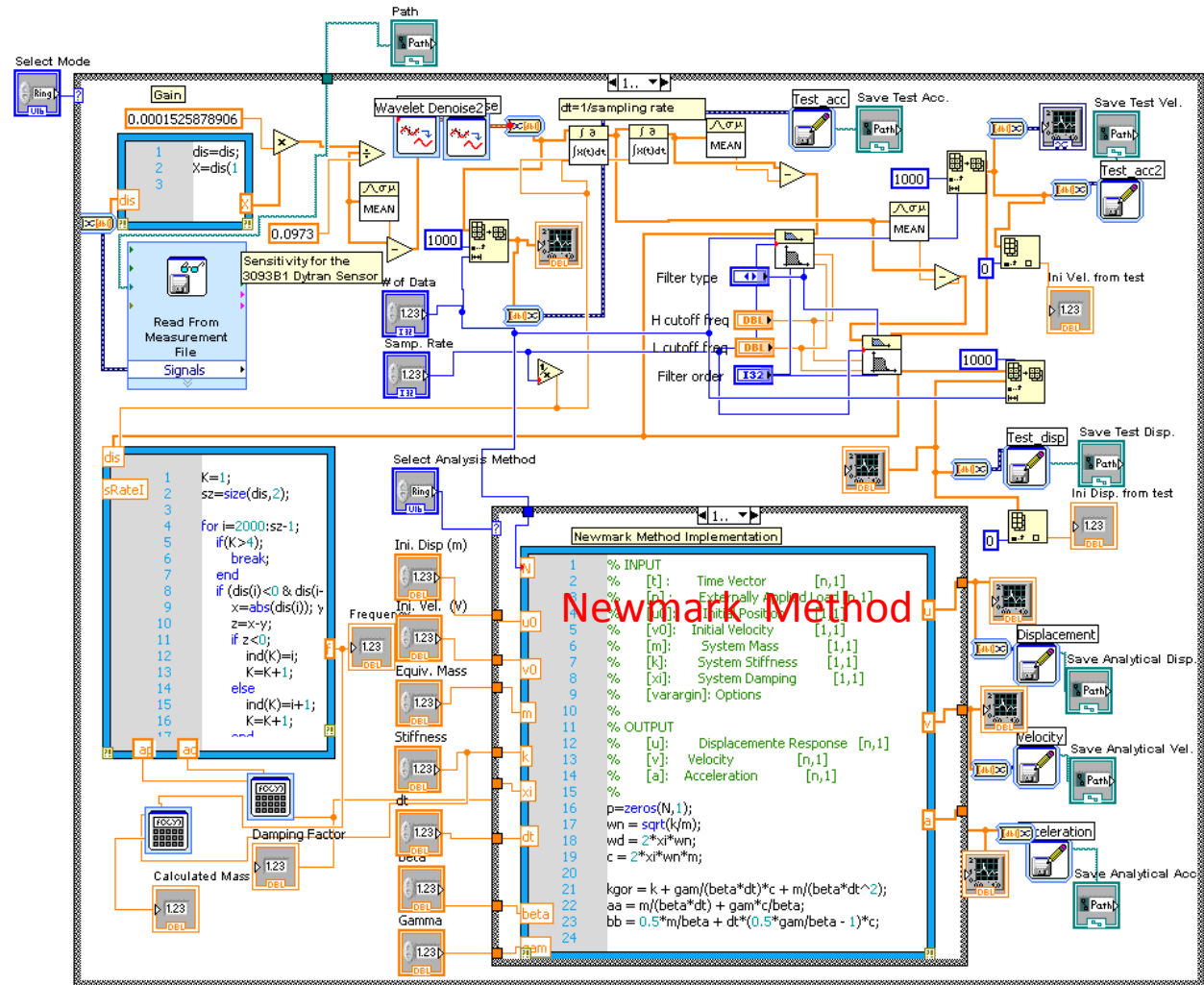
First Stack of Block Diagram of VI Program

- ❑ In the second frame, original saved raw data is read from the path for calibration and final saving.
- ❑ Wavelet denoising filter is used to remove noise before conversion to data in physical units.
- ❑ The Acc. data is converted to Vel. and Disp. data using integration
- ❑ Bandpass filter is used to pass only the frequency band of our interests.
- ❑ From the Disp. data, Init. Disp. and Vel. are automatically obtained and displayed on the front panel.
- ❑ Central Difference method and was implemented using MATALAB script to obtain numerical response of Disp., Vel. and Acc. based on the Init. Disp., Init. Vel., the effective mass, and hand-calculated stiffness
- ❑ *** Note)** Users may need to adjust the wavelet denoise.vi for the different vibration system



Second Stack of Block Diagram of VI Program

- Newmark method was implemented using math script to simulate analytical response of Disp., Vel. and Acc. based on the Ini. Disp., Ini. Vel., effective mass, stiffness, beta and gamma



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Experimental Test Results

- The stiffness is computed as

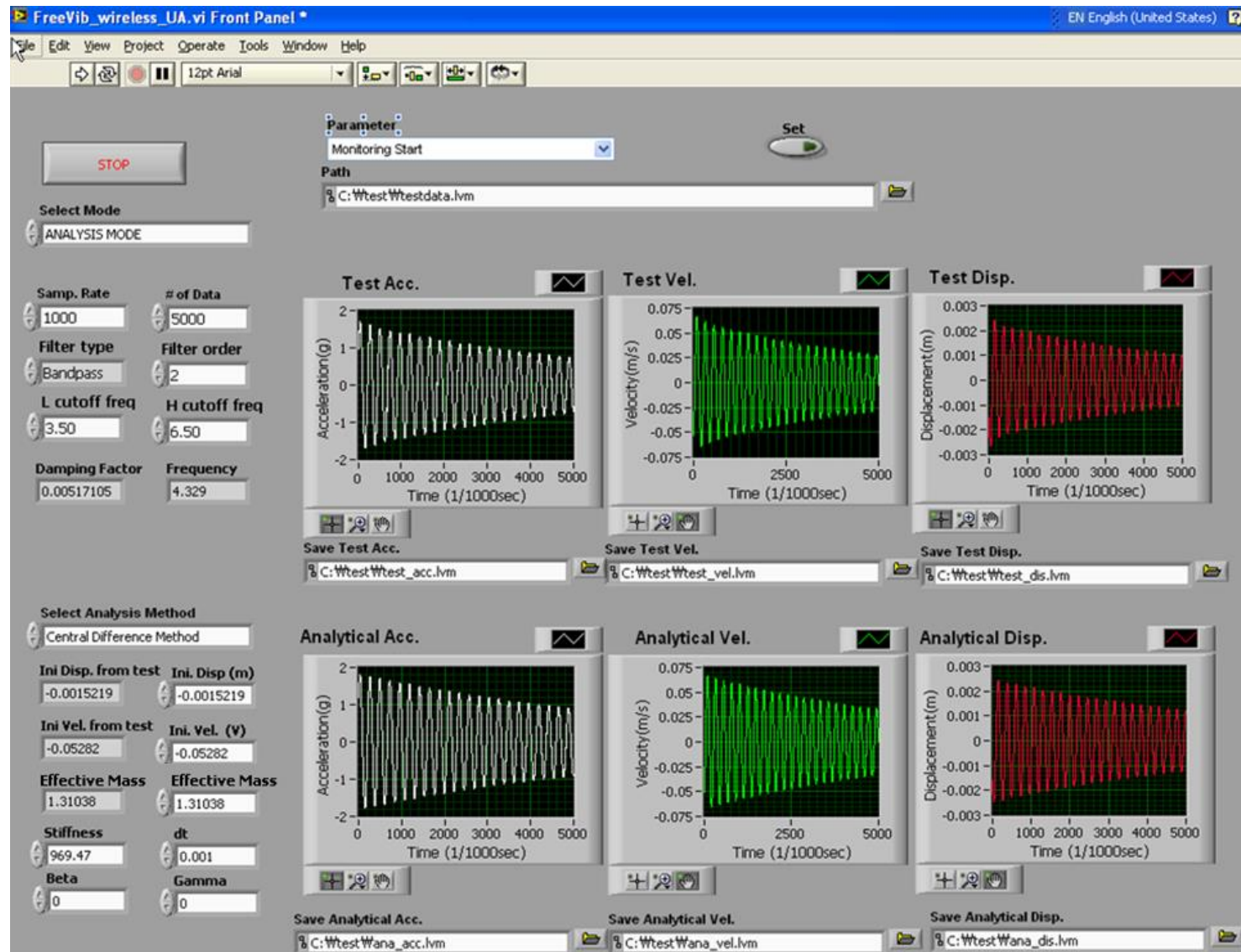
$$\frac{24EI}{L^3} = \frac{24 \times 70E9 \times (0.1082 \times 0.002^3 / 12)}{0.5^3} = 969.47 \text{ N} / \text{m}$$

- The damping factor and natural frequency were identified as 0.00517105 (less than 1%) and 4.329 Hz, respectively.
- Based on this measurements and calculation, the effective mass was estimated as

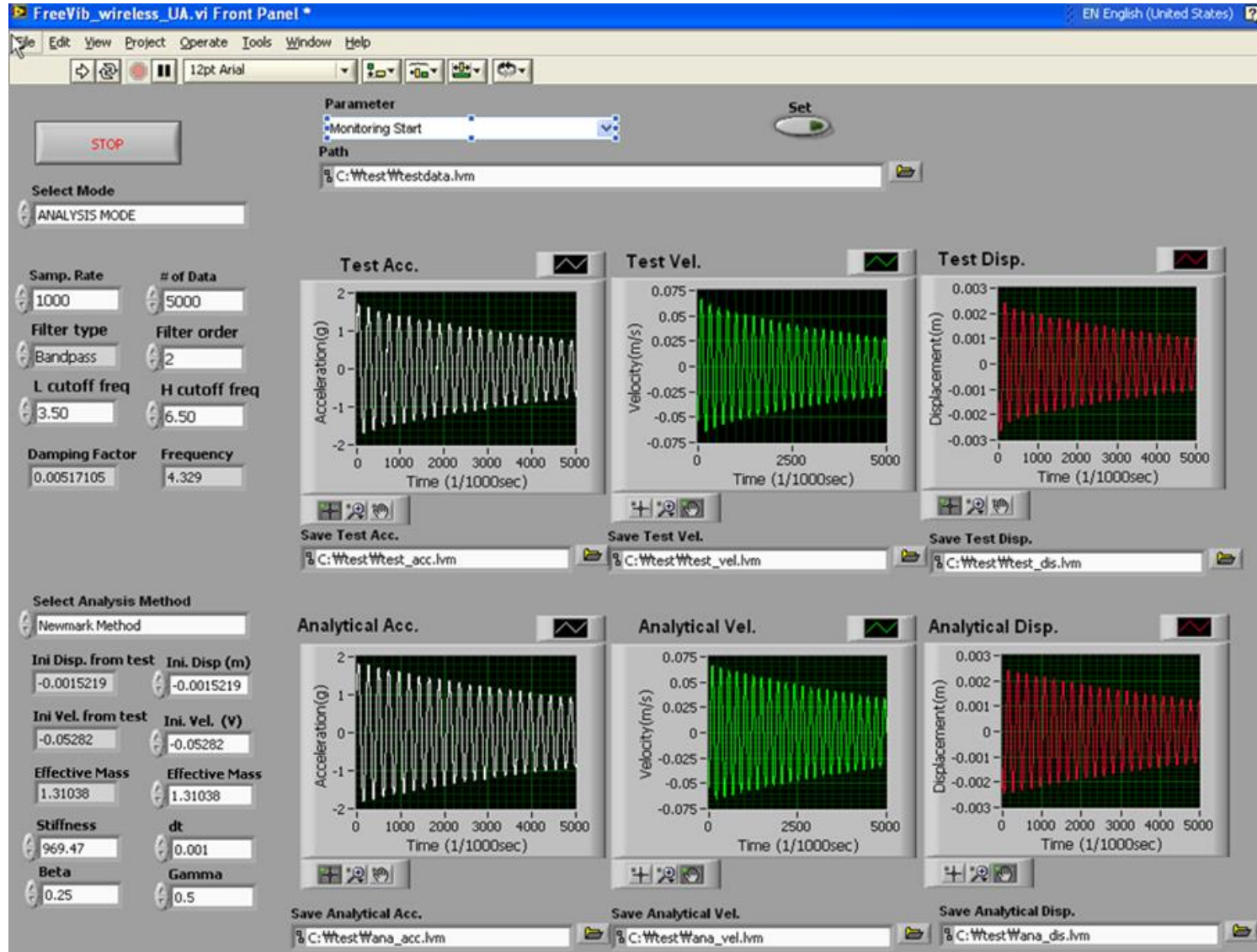
$$\frac{k}{m_{eff}} = \omega_n^2$$

$$m_{eff} = \frac{k}{\omega_n^2} = \frac{969.47}{(2\pi \times 4.329)^2} = 1.3104 \text{ kg}$$

Test Results (2)



Test Results (3)



10

References

- ❑ Roy R. Craig, Jr. and Andrew J. Kurdila, "Fundamentals of Structural Dynamics", 2nd Ed. Wiley
 - ❑ Anil K. Chopra, " Dynamics of Structures-Theory and Applications to Earthquake Engineering", 2nd Ed. Prentice Hall
-